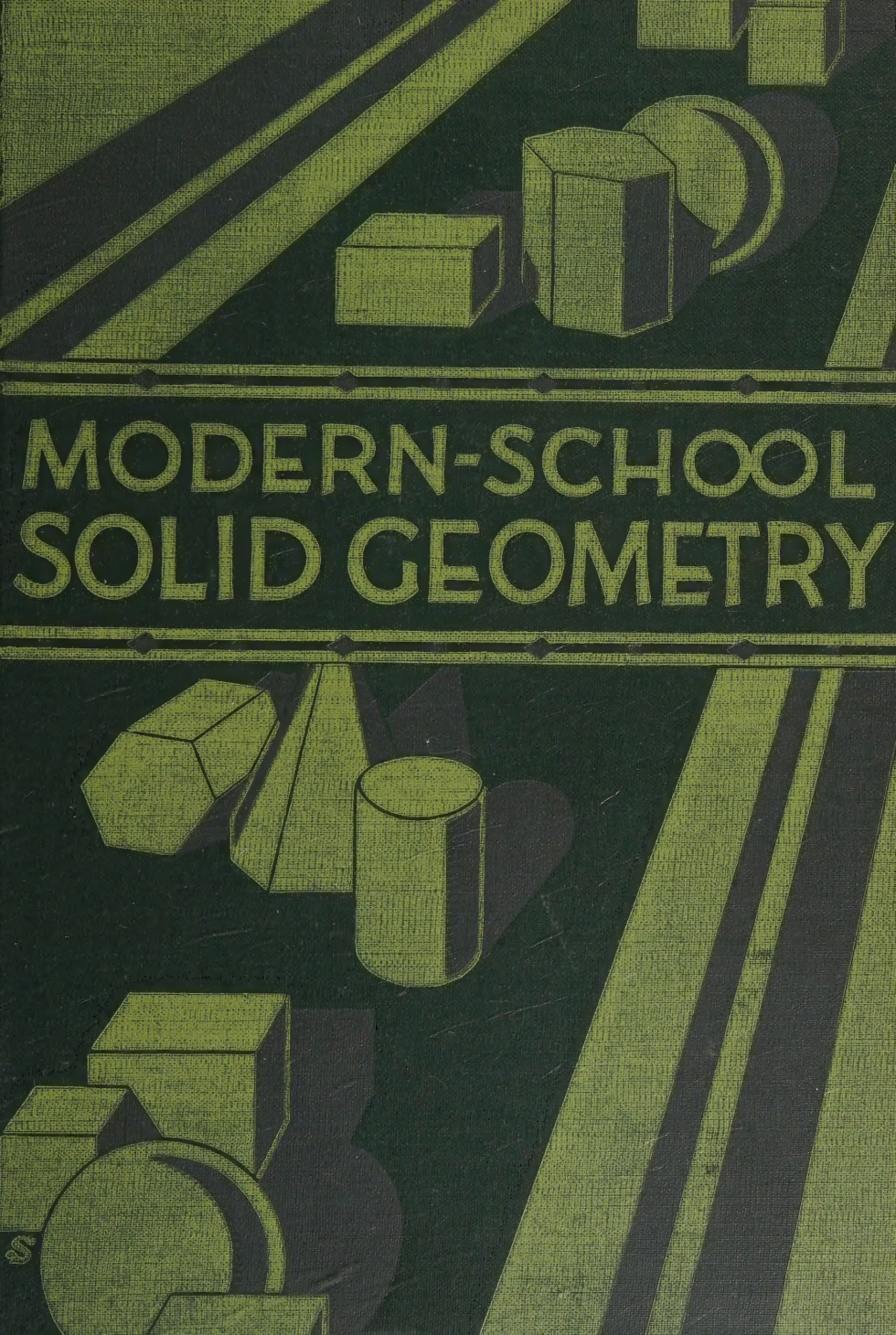
MODERN-SCHOOL
SOLID GEOMETRY

ETRY

MATH # 15

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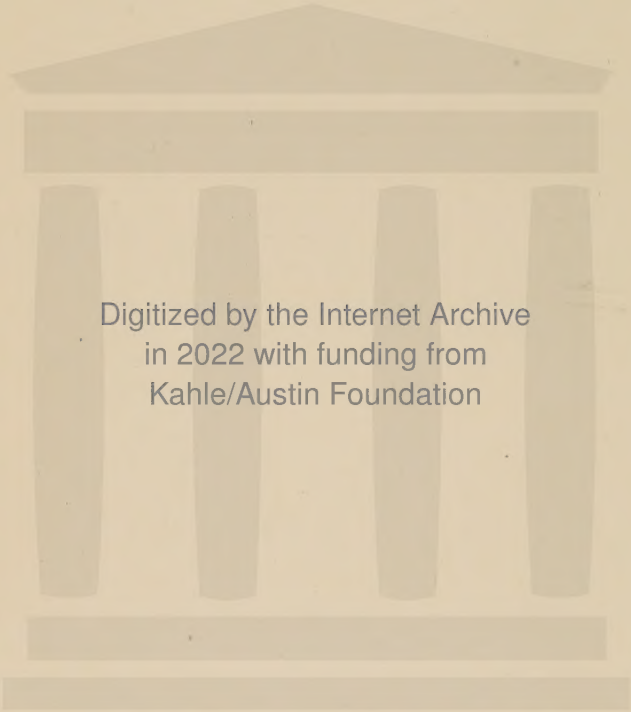
George Maycock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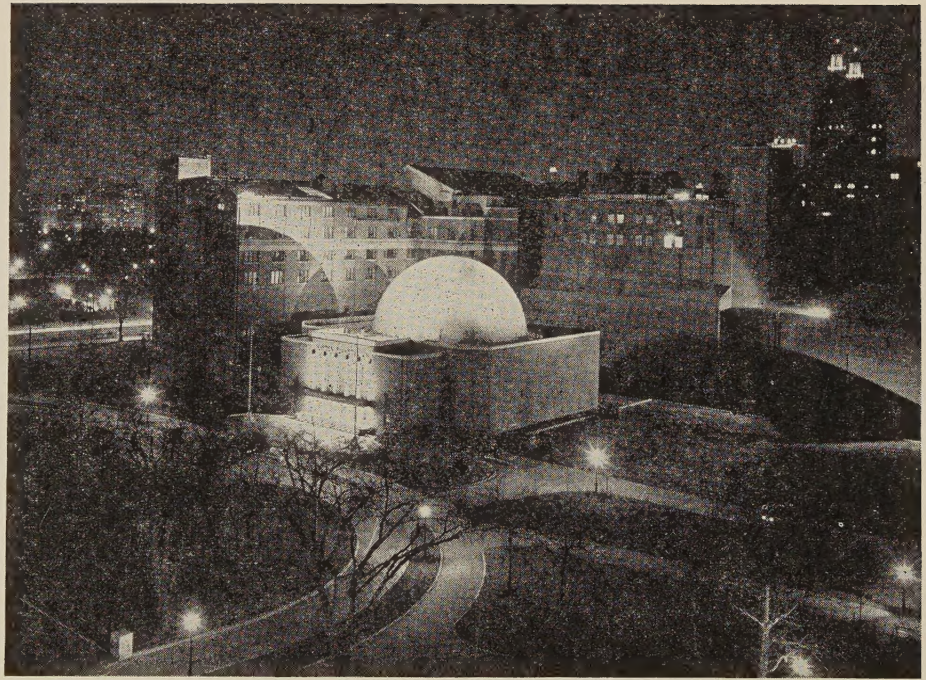

Night view of the Hayden Planetarium, New York City. *Paul J. Woolf*

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Math (synthetic geometry)

PREFACE

It is generally agreed that the primary purposes to be achieved through the study of solid geometry are (1) the discovery of geometrical truths of three-dimensional space and their establishment by logical methods, (2) the development of computational skill involving both arithmetic and literal numbers in the mensuration of common geometrical solids, (3) the practice of perceptual analysis leading to the ability to see relationships in space, and (4) an extension of understanding of the nature of logic.

Obviously a student cannot discover the truths of solid geometry or establish them by logic unless he analyzes the relationships of points, lines, and planes in space and is responsive to the same niceties of logic that are exercised in plane geometry. The clear visualization of spatial relationships is essential as the basis for understanding the postulates and definitions of solid geometry. Accordingly, the authors have given careful attention to training the student in the analysis of three-dimensional space. A brief preliminary discussion of perspective is introduced in order to acquaint students with the representation of three-dimensional figures in a plane.

For the development of computational skill the authors have included a large number of carefully graded exercises involving both arithmetic and algebraic computations.

Special features of the authors' plane geometry textbook, *Modern-School Geometry,* are continued in the present book. These include the following:

1. *Understandings are developed.* The authors are in complete accord with the view that geometry should be understood rather than memorized for reproduction. Careful inductive developments of the concepts of solid geometry are a regular instructional technique. The student is consistently required to think through his work — to learn by his own activity. Students are led by easy steps to proofs in three dimensions instead of being required at the outset to study the difficult propositions ordinarily met at the beginning of the course.

2. *Reasoning is emphasized.* The authors have recognized the fact that development in logical thinking is one of the

main objectives of the teaching of demonstrative geometry and consequently have kept the logical aspects of the course in the foreground. The student is led to appreciate the importance of definitions, assumptions, hypotheses, and conclusions. Both indirect and direct methods of proof are employed.

3. *Exercises, tests, reviews, etc., are provided.* Developmental exercises precede a new topic or proposition and prepare for it. These exercises are intended for class discussion. Proficiency exercises are placed immediately after a topic or proposition that has been developed in order to give it added meaning through use.

In addition to the developmental and the proficiency exercises there are also provided chapter tests at the end of each chapter and review exercises at intervals throughout the book. The authors have likewise retained the "Maintaining Skills" pages which proved so valuable a feature of the plane geometry. By means of these pages the basic skills of arithmetic and algebra which have been acquired in previous years are kept alive in the student's mind.

4. *Interest features are provided.* It is true that the primary interest in any class must be the intellectual one that arises from an understanding of the subject matter, but the authors have not on this account neglected to make the book as attractive as possible from other points of view. The arrangement of the pages in both type and illustration makes this textbook unusually inviting in appearance.

The authors wish to express their appreciation to the teachers of geometry who have used their earlier books and whose recommendations have proved valuable in the preparation of the present one. In particular, thanks are due to Leland W. Smith, of Classical High School, Springfield, Massachusetts, for his careful reading of the proof and his numerous suggestions.

R. R. S.
J. R. C.

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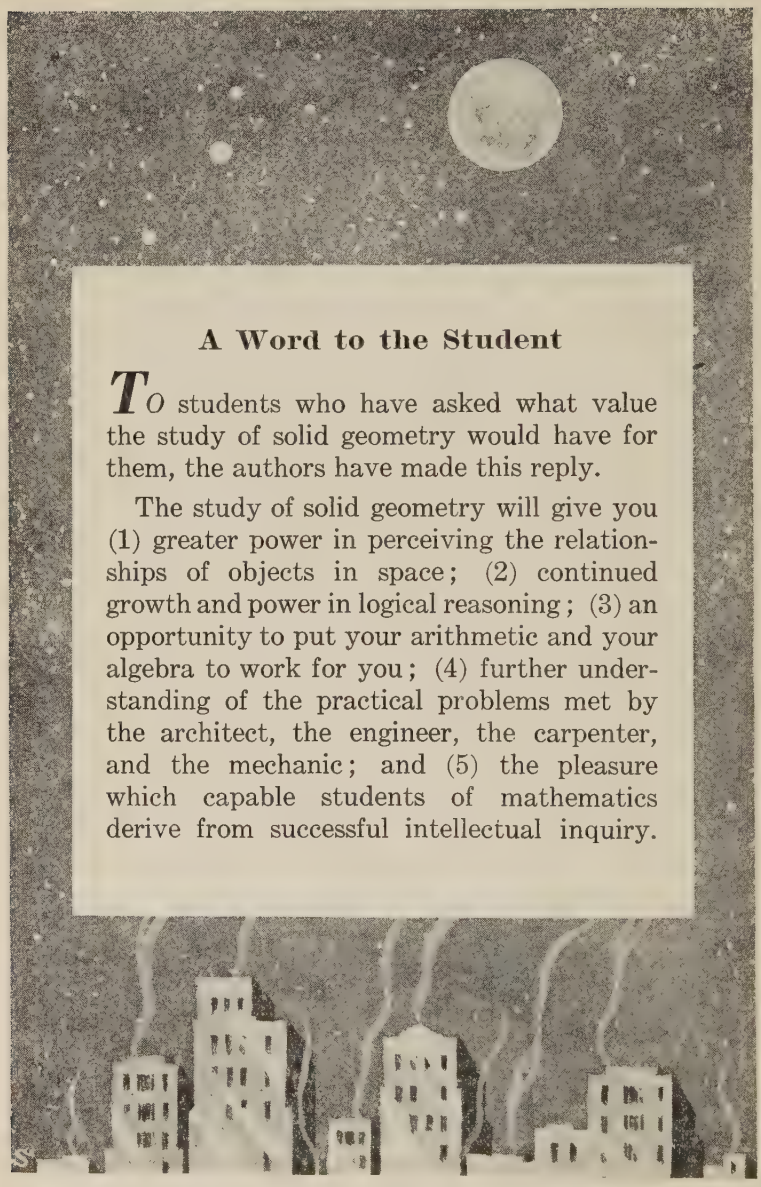


A Word to the Student

To students who have asked what value the study of solid geometry would have for them, the authors have made this reply.

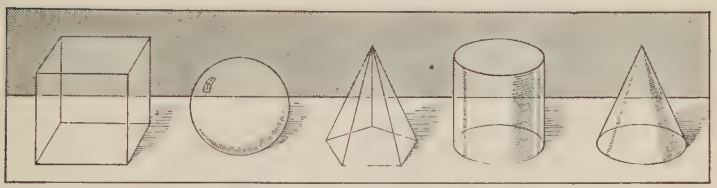
The study of solid geometry will give you (1) greater power in perceiving the relationships of objects in space; (2) continued growth and power in logical reasoning; (3) an opportunity to put your arithmetic and your algebra to work for you; (4) further understanding of the practical problems met by the architect, the engineer, the carpenter, and the mechanic; and (5) the pleasure which capable students of mathematics derive from successful intellectual inquiry.

MODERN-SCHOOL SOLID GEOMETRY

CHAPTER I

AN INTRODUCTORY CHAPTER

Three-Dimensional Figures



§ 1. In your study of plane geometry you considered for the most part figures that can be represented on a flat surface called a plane. Since such figures have only two dimensions, length and width, plane geometry is sometimes called *two-dimensional geometry*. In the study of ***solid geometry***, however, we consider figures of three dimensions: length, width, and thickness.

(1) Can you think of any material thing that has only two dimensions?[1]

(2) Does a line drawn by pencil on paper have one, two, or three dimensions?

(3) If a circle is divided into three sectors and one is painted green, another red, and the third blue, what is the color of the center of the circle?

(4) If one half of a piece of paper is painted red and the other half green, and the boundary between the two parts is a straight line, what is the color of this line?

(5) How many dimensions has this page? If the leaves of this book were torn out, would it be possible to pick up one page at a time? Explain.

[1] To the Teacher. Exercises indicated by a number in parentheses are for class discussion.

§ 2. In a geometric sense a point has no dimensions. It has *position* only. A line has only one dimension, *length*. (We represent or illustrate a geometric line by a fine pencil line.) Geometric cubes, spheres, cylinders, and other figures of three dimensions may be represented by material objects as aids to our imagination; but the real figures which we study are objects of thought only.

Although these idealized figures can exist only in the minds of those thinking about them, geometry is nevertheless a practical subject. Diagrams and material objects approximate the ideal in such a way that it is possible for carpenters, architects, engineers, and others to make definite use of the relationships discovered through the study of plane and solid geometry.

§ 3. Precise definitions of terms such as *space*, *surface*, *plane*, and *solid* cannot be given here. The meanings of these words are assumed to be familiar.

It is assumed also that you are familiar with the *rectangular solid* as a solid of six faces all of which are rectangles, with the *cube* as a solid of six faces all of which are squares, and with the terms *edges* and *vertices* in connection with these figures. Such solids as the *pyramid*, the *cylinder*, the *cone*, and the *sphere* are likewise familiar to you in a general way, but we shall not assume the knowledge of any specific properties of these solids. (See the drawings of these figures on page 1.)

§ 4. The definitions of plane geometry are used in solid geometry, without change where applicable.[1] There is sometimes, however, a change of emphasis. Parallel lines were defined as lines that lie in the same plane and will not meet, no matter how far extended in either direction. This definition of plane geometry is the same in solid geometry; but when more than one plane is to be considered, it is just as important to prove that the lines are *in the same plane* as to prove that they never meet.

[1] A list of definitions used in plane geometry is given on pages 227–230.

§ **5.** (1) Lay a straightedge on the blackboard or table top. Do all the points of the straightedge appear to touch the blackboard?

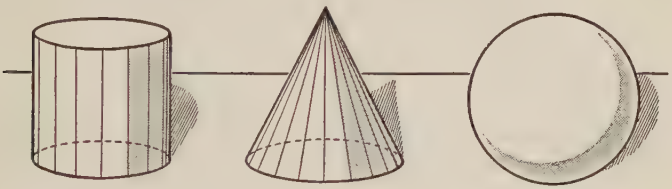
(2) Would a straight line joining any two points on the blackboard lie wholly in the plane of the blackboard?

These considerations suggest our first postulate:

POSTULATE. **A straight line joining any two points in a plane lies entirely in the plane.**

From this postulate it is easy to deduce the corollary:

§ **6.** COROLLARY. **If a straight line does not lie entirely in a plane, it can have not more than one point in common with the plane.** (Suppose it had two points.)



Straight lines drawn as indicated on the cylinder and the cone will lie entirely in the surface. Straight lines joining any two points on the surface of a sphere will have only two points in common with the surface of the sphere.

EXERCISES

1. Hold two pencils in such a way that they are not parallel but will never meet, no matter how far extended.

2. Are two lines perpendicular to the same line parallel?

3. Can you give the names of any surfaces such that a straight line connecting any two points on them will not lie entirely in the surface?

4. Can you find two points on the curved surface of a cylinder such that a straight line connecting them will lie entirely in the surface?

§ 7. Any plane must be thought of as dividing all space into two parts. It is *limitless* in length and breadth. Think of the top of a large flat table and then imagine that it is extended in all directions indefinitely. Any two points, or lines, or figures drawn on this limitless surface are said to ***lie in the same plane.*** If this table is raised or lowered or tipped, another limitless plane is determined.

(1) Two flat tables of exactly the same height stand on a level floor. Do the tops of these tables lie in the same plane?

§ 8. Since a plane indefinite in extent cannot be drawn, it is customary to indicate that portion of a plane with which we are directly concerned by enclosing it as shown in the figure. A plane is identified by naming the letters at the ends of a diagonal, as plane MN in Figure 1.

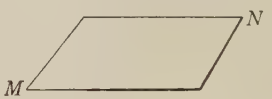


FIG. 1

§ 9. If a line lies in a plane, the plane is said to ***contain*** the line. ***To pass a plane through a line*** is to construct a plane in such a way that the line lies wholly in the plane. Similarly, if a point lies in a plane, the plane is said to contain the point. To pass a plane through a point is to construct a plane which contains the point.

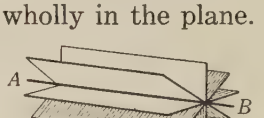


FIG. 2

§ 10. POSTULATE. **Through a given line an unlimited number of planes may be passed.** (See Figure 2.)

It is well to recall here a few things that you learned in your study of plane geometry: A ***postulate*** is a proposition — i.e., a statement of a relationship — that is assumed. A ***theorem*** is a proposition that is proved. A ***geometric proof*** is a series of statements, each of which *must* follow from statements that have preceded it. In the development of a deductive science all reasons in the proof of a theorem must be reasons that have *previously* been accepted. The proof of the *first* theorem, at least, must depend upon statements that have not been proved.

Determining a Plane

Certain conditions are said to ***determine*** (the position of) ***a plane***, if there is one plane and only one that fulfills these conditions.

The following exercises will help you discover what the conditions are which determine a plane.

(1) Three pins not in a straight line project unequal distances through a piece of cardboard. Can you place a piece of stiff cardboard so that it will rest on all three of these points?

(2) Four pins project unequal distances through a piece of cardboard. Two pins project 2 inches, one pin projects 1 inch, and the fourth pin projects 3 inches. Can you, in general, place a piece of stiff cardboard so that it will rest on all four points?

(3) Hold a straightedge and pencil point in such a way as to represent a straight line and a point outside it. Can you place a piece of stiff cardboard on the point of the pencil and the straightedge so that the cardboard will lie flat on the straightedge? Can you do the same thing with the straightedge and pencil point in any position?

(4) Hold two straightedges in such a way as to represent two intersecting lines. Note that a piece of stiff cardboard will lie flat on both straightedges. Will this be true if the straightedges meet each other at any angle?

(5) Hold two straightedges so that they are parallel to each other. Lay a piece of stiff cardboard across them. Note that the cardboard lies flat on both of the straightedges.

(6) Can you think of any kind of triangle that does not lie entirely in one plane?

(7) Can you think of any kind of quadrilateral that does not lie entirely in one plane?

These exercises suggest the postulate and corollaries stated on page 6.

§ 11. POSTULATE. **Three points not in the same straight line determine a plane.**

This postulate may also be stated: **Through three points, not in the same straight line, one plane, and only one, can be passed.** (Fig. 1.)

§ 12. COROLLARY. **A line and a point not on the line determine a plane.**

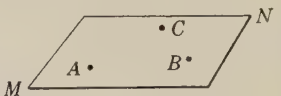


FIG. 1

Any two points of the line and the point not on the line determine a plane. The whole line will lie in the plane. Why? This can be visualized by thinking of a plane revolving about line AB until it contains C. (Fig. 2.)

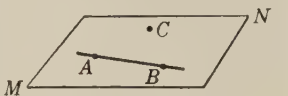


FIG. 2

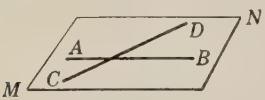


FIG. 3

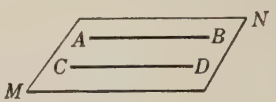


FIG. 4

§ 13. COROLLARY. **Two intersecting lines determine a plane.** Why? (Fig. 3.)

§ 14. COROLLARY. **Two parallel lines determine a plane.**

The two parallel lines lie in one plane by definition. If they could lie in two planes, then there would be two planes through one of the lines and any point of the other, which is impossible according to the first corollary above. (Fig. 4.)

EXERCISES

1. Thus far in this chapter, what propositions have been stated as postulates? as corollaries?

2. Explain why an object with three legs, such as the standard for a transit, always rests firmly on the floor or ground, while an object with four legs, such as a chair or a table, may not.

3. How does a plasterer test the flatness of a wall surface?

4. Describe the method used by a workman in constructing a concrete walk so that it will be flat. What geometric principles does he use?

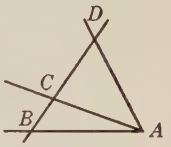
5. How many different planes are determined by four points not all in the same plane?

6. How many different planes are determined by three parallel lines not all in the same plane?

7. If three lines pass through a point, will they always lie in one plane? How many planes may they determine?

8. Is a triangle necessarily a plane figure? Explain.

9. Three lines pass through a given point and intersect a fourth line, which does not pass through this point. Show that all four lines lie in the same plane.



(Ex. 9)

10. Prove: Four points do not necessarily lie in one plane.

11. Is a quadrilateral necessarily a plane figure? Illustrate your answer by creasing on a diagonal a rectangular piece of paper.

12. When a point moves, it is said to ***generate*** a line. Points A and B are in the plane MN. Point C moves in the plane but is always equally distant from A and B. What line is generated?

13. A straight line which moves in any way except along itself generates a surface. A moving line AB always passes through the point P and always contains some point of another line CD, which does not pass through P. What is the nature of the surface generated?

14. A line AB moves so that it is always parallel to its former positions and always contains some point of another line CD. What is the nature of the surface generated?

15. Prove: If two lines are parallel, a line joining any point of one with any point of the other lies in the plane of the parallel lines.

16. Prove: A parallelogram is necessarily a plane figure.

Comprehension Test

1. In the study of solid geometry, we consider ___?___ dimensions, ___?___, ___?___, and ___?___.

2. Any material thing has ___?___ dimensions.

3. A line drawn by pencil on paper has ___?___ dimension(s).

4. A point, in the geometric sense, has ___?___ dimension(s).

5. A line, in the geometric sense, has ___?___ dimension(s).

6. Are two lines necessarily parallel if they never meet, no matter how far extended? Answer *yes* or *no*.

7. Are two lines perpendicular to the same line necessarily parallel? Answer *yes* or *no*.

8. A straight line joining ___?___ ___?___ ___?___ in a plane ___?___ ___?___ in the plane.

9. A straight line that does not lie entirely in a plane will intersect the plane in ___?___ ___?___ ___?___ ___?___ point.

10. A plane divides all space into ___?___ ___?___ and is ___?___ in length and breadth.

11. If a plane is passed through a line, then the line ___?___ ___?___ in the plane.

12. Through a given line an ___?___ ___?___ of ___?___ may be passed.

13. If a line lies in a plane, the plane is said to ___?___ the line.

14. Does a line connecting any two points on a sphere lie entirely in the surface of the sphere? Answer *yes* or *no*.

15. A postulate is a ___?___ that is ___?___.

16. A plane is determined by (*a*) ___?___, (*b*) ___?___, (*c*) ___?___, (*d*) ___?___.

Maintaining Skills[1]

(*Exercises on triangles*)

1. The arms of a right triangle are 7 and 5 inches, respectively. Find the length of the hypotenuse to three significant figures.

2. In $\triangle ABC$, $\angle B$ is a right angle and $BD \perp AC$. Evaluate all the sides and AD and DC in the following cases:

(*a*)	$AD = 4$	$DC = 9$	(*e*)	$AB = 6$	$AC = 10$
(*b*)	$AD = 3$	$DC = 4$	(*f*)	$AB = 7$	$AC = 12$
(*c*)	$AB = 5$	$AD = 3$	(*g*)	$AB = 5$	$BC = 12$
(*d*)	$AB = 8$	$AD = 6$	(*h*)	$AB = 4$	$BC = 5$

3. In $\triangle ABC$, $\angle B$ is a right angle and $\angle A = 60°$. If AB is 5 inches long, how long are AC and BC? If AC is 7 inches long, how long are AB and BC? If BC is 8 inches long, how long are AB and AC?

4. In $\triangle ABC$, $\angle B$ is a right angle and $\angle A = 60°$. If AB is a units, how long are AC and BC? If AC is b units, how long are AB and BC? If BC is c units, how long are AB and AC?

5. In $\triangle ABC$, $\angle B$ is a right angle and $\angle C = 30°$. If $BC = 5$ inches, what is the area of the triangle?

6. Each side of an equilateral triangle is 8 inches. What is its area?

7. Each side of an equilateral triangle is a inches. What is its area?

8. The altitude of an equilateral triangle is 12 inches. What is its area?

9. The altitude of an equilateral triangle is a inches. What is its area?

10. The three altitudes of an equilateral triangle are drawn. If the length of an altitude is $4\sqrt{3}$ units, how far is it from any vertex to the intersection of the altitudes?

[1] To the Teacher. The work on pages headed "Maintaining Skills" is for the purpose of keeping alive the skills in arithmetic, algebra, and plane geometry which pupils have gained in previous years. They should be used according to the needs of the class.

Perspective

§ **15.** The figures of plane geometry can be represented very easily on paper or on a blackboard, since they are two-dimensional. It is not easy, however, to represent on a flat surface the three-dimensional figures of solid geometry. When three-dimensional figures are represented on a flat surface as they appear to the eye, they are said to be drawn ***in perspective.***

Ewing Galloway

Park Avenue, New York City. Note how the parallel lines seem to converge toward a point as they recede from the observer.

The drawing of neat, clear diagrams to represent figures that occupy space is an aid in visualizing the relationships of lines and planes. Rulers should be used to draw straight lines and compasses to draw circles. Compasses and rulers,

however, cannot be used in construction work as they were in plane geometry. A circle in a horizontal plane directly in front of the observer cannot be drawn with compasses, for it must be represented as a straight line. A line represented as receding from the observer cannot be bisected by the use of compasses, for equal distances seem to become shorter as they get farther away.

(1) Close your book and hold it in a horizontal position in such a way that you are looking down at it. Keeping the book in a horizontal position, raise it slowly until it is directly opposite your eyes. Note that the cover seems to grow narrower, from front to back, until only the edge of the book can be seen. Raise the book still higher. Note that the lower cover can now be seen and seems to grow wider as the book is raised.

(2) Do a similar exercise with a 50-cent piece or a watch. Note the appearance of the circle in different positions.

(3) Place a book face downward on your desk so that it will be in a horizontal position directly in front of you, with its back edge parallel to the front edge of the desk. Slip a string under the cover along the back edge. Hold both ends of the string vertically over the back edge of the cover as in

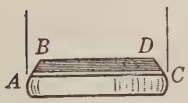


FIG. 1

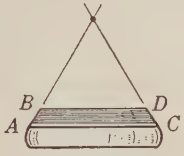


FIG. 2

FIG. 3

Figure 1. Close one eye and move the two ends of the string until they hide the edges *AB* and *CD* of the cover. (Fig. 2.) Be sure to hold the ends of the string in a vertical plane over the back edge of the cover, not letting them fall backward or forward. Raise the book a few inches, keeping the book level and the string taut. In order to make the string hide the edges of the book, the intersection of the ends must be brought nearer to the book, as in Figure 3.

If the book is raised higher, the intersection of the ends must be brought still nearer to it. Move the book to your right and to your left, keeping it always in a horizontal position, and try the same experiment. You will note that, when the ends of the string hide the edges of the cover, they intersect always at a point on a level with your eye. It is evident from this experiment that a person drawing this book must represent the receding edges of the cover as lines converging toward a point directly in front of him.

(4) The following three diagrams are cubes drawn in perspective. Figure 4 represents a cube a little below and to the left of the eye of the observer. Figure 5 represents a cube a little below the eye of the observer, with the edge *AD*

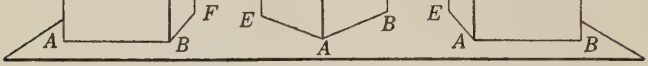


FIG. 4 FIG. 5 FIG. 6

directly before him. Figure 6 represents a cube a little below and to the right of the eye of the observer. Measure the angle *BCG* in Figure 4. What size is this angle in the actual cube? Measure *AD*, *BC*, and *FG*. Also measure *AB*, *DC*, and *HG*. Note that equal lines, such as *BC* and *FG* or *AB* and *HG*, are not drawn equal, since equal distances seem to become shorter as they get farther away from the observer. Note also that parallel lines, such as *BF* and *CG*, that recede from the observer seem to converge toward a point. Test these facts in the other figures.

(5) Bring to class pictures of buildings or of the interiors of buildings in which there are several geometric figures. Measure the angles that are *actually* right angles in the buildings. Are parallel lines actually parallel in the picture? Are the circles pictured as circles?

(6) Can you think of a method of finding the center of a circle inscribed in a square drawn in perspective?

§ 16. Figure 1 shows why objects at a distance look smaller than they really are. The heights of objects are judged by comparing them with other objects. When a man at *A* looks at the top of a pole *BE*, he cannot see the top of the pole *FC*. Since he has to lower his eyes to see *F*, it appears lower than the top of the pole *BE*. Similarly, *D* appears lower than *F*. Likewise, since he has to raise his eyes as he looks first at the bottom of the pole *BE* and then at *C* and *G*, *C* seems higher than *E* and *G* higher than *C*. *DG* and *FC*, then, as they are compared with *BE*, will look to be only the height of *TV* and *RS*, respectively. It is also evident that, since *F* and *D* look lower than *B* and since *C* and *G* look higher than *E*, the two parallel lines *BD* and *EG* will seem to converge toward a point. To the man at *A*, the poles will look similar to those in Figure 2.

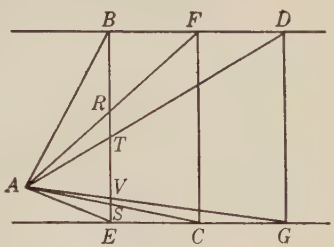


FIG. 1

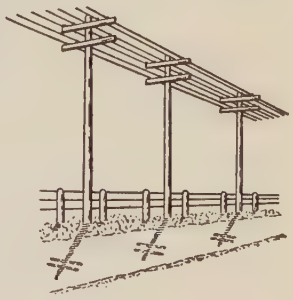


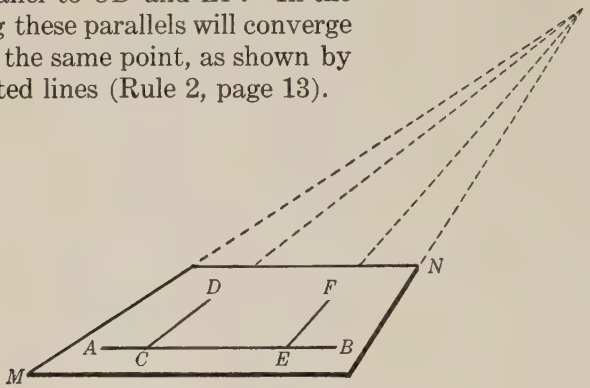
FIG. 2

Diagrams in solid geometry are not usually drawn strictly according to the rules of perspective. Instead they are drawn conventionally so that all those studying the subject may understand the figures.

The following rules will help you in making drawings:

1. Horizontal lines at right angles to the line of vision appear parallel.
2. Parallel lines that recede from the observer seem to converge toward a point on a level with the eye.
3. Lines nearer the observer should be made darker than lines farther away.
4. Lines that cannot be seen in the actual object should be drawn as dotted lines in a figure.

The figure below represents a line AB drawn in a horizontal plane MN, with DC and EF in the plane and perpendicular to AB. A portion of the horizontal plane is represented by a rectangle drawn in perspective. If AB is drawn parallel to one side of the rectangle, the adjacent sides of the rectangle are parallel to CD and EF. In the drawing these parallels will converge toward the same point, as shown by the dotted lines (Rule 2, page 13).



EXERCISES

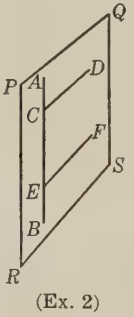
1. Represent a portion of a horizontal plane MN with line AB in the plane.

(a) Represent a line parallel to AB in plane MN.

(b) Represent a line perpendicular to AB in plane MN.

(c) Represent a line in plane MN parallel to the line you drew in (b).

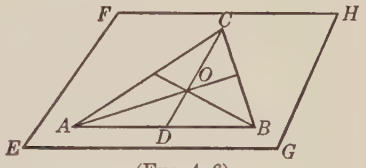
2. Represent a portion of a vertical plane RQ with line AB in the plane as in the figure. Draw lines as in (b) and (c) of Exercise 1. PQ, CD, EF, and RS if extended will meet at the same point.



(Ex. 2)

3. A and B are points in a horizontal plane each equally distant from points C and D in the same plane. Prove that AB is the perpendicular bisector of CD. Draw the figure carefully, according to the rules on page 13.

4. Represent an equilateral triangle in a horizontal plane. Draw the three medians of the triangle. (First draw AB parallel to EG. Then draw the median DC. EF, DC, and GH will, if extended, intersect at the same point. See the figure.)



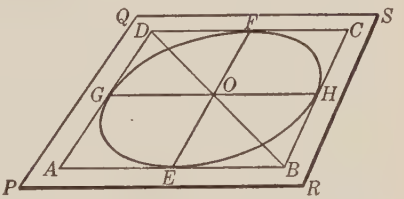
(Exs. 4–6)

5. In the figure for Exercise 4, if AB is 5 inches, how long is OD? If AB is a inches, how long is OD?

6. In the figure for Exercise 4, if OC is $2\sqrt{3}$ inches, how long is AB? If OC is a inches, how long is AB?

7. Represent an equilateral triangle in a horizontal plane. Draw the three altitudes of the triangle. If one of the altitudes is $4\sqrt{3}$ inches, what is the length of each side? What is the length of the line from a vertex to the point of intersection of the altitudes? Draw the figure carefully. Then draw the figure as if it were in the plane of your paper in its correct proportions.

8. Represent a square in a horizontal plane. Inscribe a circle in the square. The center of the circle is found by drawing the diagonals of the square. PQ, EF, and RS will intersect at the same point (Rule 2, page 13). PR, AB, GH, DC, and QS are parallel (Rule 1, page 13). E, F, G, and H will be the points of tangency of the circle. Explain.



(Exs. 8–9)

9. If, in the figure for Exercise 8, AB is 10 in., what is the area of the inscribed circle?

10. Triangle ABC is a right triangle lying in a horizontal plane. CD is the altitude upon the hypotenuse drawn from the vertex of the right angle C. Angle A is 30° and AB is 10 inches. Evaluate all the lines and angles of the figure. Draw the figure carefully. (See § 26.)

Intersection of Surfaces

§ 17. The *intersection of two surfaces* is the locus of points common to the two surfaces.

§ 18. POSTULATE. **If two planes have one point in common, they must have at least two points in common.**

§ 19. COROLLARY. **The intersection of two planes is a straight line determined by any two points which the planes have in common.**

It is necessary to show that the line determined by the two points is the locus of points common to the two planes. What conditions must be fulfilled?

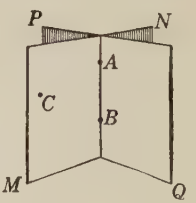
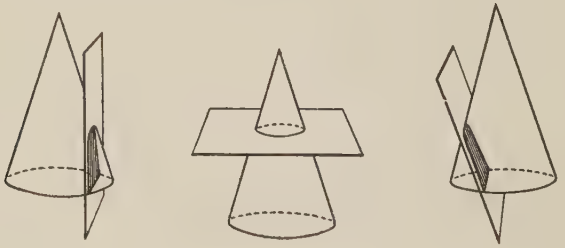


FIG. 1

If MN and PQ intersect, they must have at least one point A in common. If they have A in common, then some point B must also be common to both planes. Why? Then all points of line AB are common to both planes. Why? Furthermore, any point not on AB cannot lie in both planes. Why? Hence all points common to both planes lie in AB. AB is, therefore, the locus of such points.

§ 20. COROLLARY. **If a line lies in each of two planes, it is their intersection.**

§ 21. **Conic sections.** A section formed by the intersection of a plane and a cone is called a *conic section*. The figures below show the intersections of a cone and a plane cutting it in three different ways.



§ 22. When a straight line intersects a plane, the point of intersection with the plane is called the *foot* of the line. B is the foot of line AC in plane MN (Fig. 1).

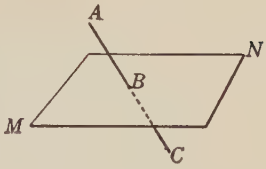


FIG. 1

§ 23. A straight line is said to be ***perpendicular to a plane*** if it is perpendicular to every line that may be drawn in the plane through its foot. If a line is perpendicular to a plane, the plane is said to be ***perpendicular to the line*** (Fig. 2). (If a line is perpendicular to a plane, it is perpendicular to every line in the plane through its foot.)

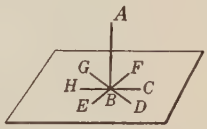


FIG. 2

(1) What is the definition of perpendicular lines in plane geometry?

(2) How many lines can be drawn perpendicular to a given line at a given point on the line?

§ 24. When a straight line intersects a plane but is not perpendicular to *every* line that may be drawn through its foot in the plane, it is said to be ***oblique*** to the plane.

EXERCISES

1. Draw several lines through a given point O on a piece of paper. Letter the lines. Place a pencil AO so that it is perpendicular to all these lines. Read the various right angles.

2. Three lines AB, CD, and EF are drawn in plane MN through O. At O a line HO, not in plane MN, is drawn perpendicular to AB but oblique to CD and EF. Of the six angles formed by HO and each of the lines, ___?___ are right angles, ___?___ are ___?___ angles, and ___?___ are ___?___ angles.

3. If a line is perpendicular to each of two lines in a plane, is it perpendicular to the plane of the lines?

4. Draw a figure to illustrate Exercise 3 when the line is not perpendicular to the plane; when the line is perpendicular to the plane.

Drawing Three-Dimensional Figures

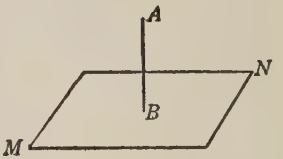
§ 25. In order to introduce at this point a few simple exercises which will give practice in drawing figures of three dimensions and some skill in visualizing space relationships, the fact that *one and only one line can be drawn perpendicular to a plane at any point in the plane* will be assumed for the present. It is important that the figures for the following exercises be drawn carefully.

EXERCISES

1. Represent a line perpendicular to a horizontal plane.

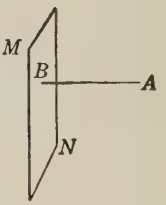
2. Represent a line perpendicular to a vertical plane.

3. Represent two intersecting lines in a horizontal plane. At their intersection represent a line perpendicular to the plane.



(Ex. 1)

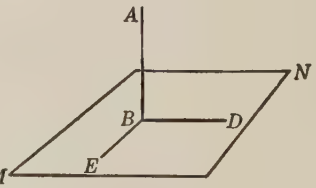
4. Represent a line perpendicular to a horizontal plane, and through its foot draw in the plane two lines making right angles with each other, as in the figure for this exercise. Letter your drawing the same as the figure and read all the right angles. Why are they right angles?



(Ex. 2)

5. If, in the figure for Exercise 4, AB is 12 inches, and EB and BD are each 5 inches, how long are AE and AD? How long is ED? Find the area of $\triangle AED$.

6. AB is perpendicular to plane MN at B. BC and BD are in plane MN. If BC, BD, and AC are 3, 2, and 5 inches respectively, how long is AD?



(Exs. 4–5)

7. If, in Exercise 6, BC, BD, and AC are a, b, and c units respectively, how long is AD?

8. What is the area of $\triangle ABD$ in Exercises 6 and 7?

9. AB is perpendicular to plane MN at B. BC and BD are two equal lines in MN. Prove in good form that $\triangle ABC$ and $\triangle ABD$ are congruent right triangles.

10. Prove that equal oblique lines drawn from a point to a plane meet the plane at equal distances from the foot of the perpendicular from the point to the plane.

11. Prove that oblique lines drawn from any point on a perpendicular to a plane, meeting the plane at equal distances from the foot of the perpendicular, are equal.

12. At the intersection of the diagonals of a square drawn in a horizontal plane a line is drawn perpendicular to the plane. Prove that lines drawn from any point on the perpendicular to two opposite vertices are equal. Also prove that lines drawn from any point on the perpendicular to two adjacent vertices are equal. (See § 26.)

13. At the center O of the square $ABCD$ a line HO is drawn perpendicular to the plane of the square. If AB and HO are 6 and 8 inches respectively, how long is HA?

14. If, in Exercise 13, AB and HO are a and b units respectively, how long is HA?

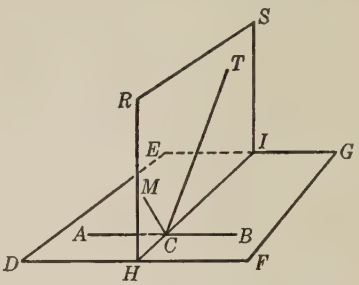
15. What is the area of $\triangle AHC$ in Exercises 13 and 14?

16. At the intersection O of the altitudes of an equilateral triangle ABC, a line DO is drawn perpendicular to the plane of the triangle. If AB and DO are 12 and 4 inches respectively, how long is DA?

17. AB is a line perpendicular to plane MN and intersects it at point O. Point O is the middle point of AB. C and D are two points in plane MN. Prove that CO and DO are both perpendicular bisectors of AB.

18. Plane MN is perpendicular to AB at its middle point C. P is any point in plane MN. Prove that $AP = PB$. Draw the figure carefully.

19. Represent a horizontal plane with line AB in the plane. Represent another plane intersecting the first plane and perpendicular to AB at C, the midpoint of AB. Draw two lines CM and CT in the perpendicular plane. DE, HI, FG, and RS will, if extended, intersect at the same point.



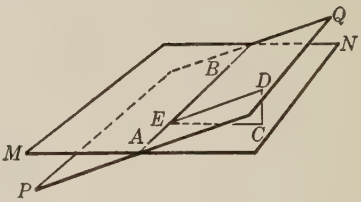
(Ex. 19)

(*a*) Name all the right angles in the figure.

(*b*) Prove that $AM = MB$.

(*c*) If AC, HC, MC, TC, and CI are 5, $3\frac{1}{2}$, 4, 8.7, and $6\frac{3}{4}$ units respectively, what are the areas of triangles AHB, AMB, ATB, and AIB?

20. Represent a horizontal plane with line AB in it. Through AB pass a plane that is neither horizontal nor vertical. From any point C in the first plane represent a line perpendicular to the first plane and intersecting the second plane in D. From D represent a line DE perpendicular to AB. Draw EC. If $\angle DEC$ is 30° and EC is 10 units, how long are ED and DC?



(Ex. 20)

21. AE is perpendicular to plane MN at B; B is the midpoint of AE; BD and BC are equal lines drawn in plane MN; $BD \perp BC$.

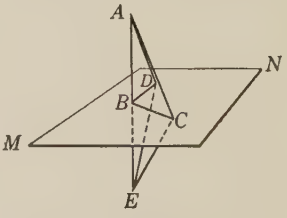
(*a*) Is DC in plane MN?

(*b*) Are AD, DE, and AE all in one plane?

(*c*) Are DC, CE, and DE all in one plane?

(*d*) Are AD, DE, and DC all in one plane?

(*e*) Prove that $\triangle ADC \cong \triangle DEC$. Prove that $\triangle ADE \cong \triangle ACE$.



(Ex. 21)

Chapter Test: Part I

1. How many dimensions has (*a*) a point? (*b*) a straight line? (*c*) a plane? (*d*) a cube?

2. The materials of geometry are undefined terms, defined terms, assumed propositions, and proved propositions. Give three illustrations of each from this chapter.

3. What is meant by *passing a plane through a line?* How many planes may be passed through a line?

4. State four ways of determining a plane.

5. How many planes are determined by five points, no four of which are in the same plane?

6. Give the names of two quadrilaterals that are necessarily plane figures.

7. Is a plane determined by any two transversals to two parallel lines? Explain.

8. Define *line perpendicular to a plane.*

9. The lines AB and CD lie in a horizontal plane MN and intersect at the point O. EO is perpendicular to MN at O. Lines are drawn from E to D, from E to B, and from B to D. Draw the figure. Name the angles opposite ED and EB in $\triangle EBD$. If $OD = OB$, what triangles are congruent?

10. AB is perpendicular to the plane MN at point C. $AC = CB$. CD and CE are equal lines in plane MN. Lines AE, AD, BE, BD, and ED are drawn. Draw the figure. Name four right triangles. Name four isosceles triangles. Name the angles opposite BE and BD in $\triangle BDE$.

11. Represent a horizontal plane MN with line AB in it. Represent another plane PQ perpendicular to AB at its midpoint C. PQ intersects MN in line DE. Represent a line CF in plane PQ and perpendicular to line DE. What relation has CF to AB?

Chapter Test: Part II

1. In a certain right triangle one arm is twice as long as the other. The hypotenuse is 10 inches long. How long is each of the arms? State the theorem that applies.

2. In a right triangle one acute angle is 30° and the adjacent arm is 4 inches. How long is the hypotenuse? State the theorems that apply.

3. In triangle ABC, $\angle C$ is a right angle, CD is the altitude upon the hypotenuse, and $\angle B$ is 30°. If CD is 2 inches, how long is AC? State the theorems that apply.

4. Each side of an equilateral triangle is 5 inches. How long is each of the altitudes? How far is it from each vertex to the intersection of the altitudes? State the theorems that apply.

5. AO is perpendicular to plane MN at O. OB and OC are two equal lines in plane MN. Prove that $\angle ABC = \angle ACB$.

6. If, in Exercise 5, $AO = 6$, $OC = 5$, and $\angle BOC$ is a right angle, what is the area of $\triangle ABC$?

7. If, in Exercise 5, $AO = a$, $OC = b$, and $\angle BOC$ is a right angle, what is the area of $\triangle ABC$?

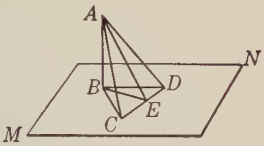
8. The plane MN is perpendicular to line AC at its midpoint B. BD and BE are lines in plane MN. Prove that $\triangle AED \cong \triangle CDE$.

9. Triangle ABC is an equilateral triangle in horizontal plane MN. At the intersection O of the altitudes of $\triangle ABC$ a line DO is drawn perpendicular to MN. If AB is 8 and DO is 6, what is the area of $\triangle ACD$?

10. If, in Exercise 9, AB is a units and DO is b units, what is the area of $\triangle ACD$?

Miscellaneous Exercises

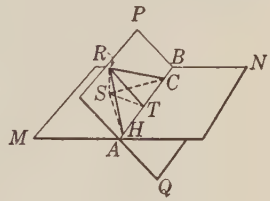
1. AB is perpendicular to plane MN at B. CD is a line in plane MN not passing through B. BE is perpendicular to CD at E. $CE = ED$, $AB = 24$ units, $BC = 10$ units, and $CE = 6$ units. Find the length of AE. Test to see if $\angle AEC$ is a right angle by using the fact that, if $a^2 + b^2 = c^2$ in any triangle, it is a right triangle.



(Ex. 1))

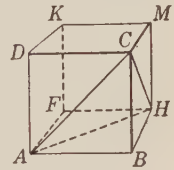
2. Two planes MN and PQ intersect each other in the line AB. Is AB a straight line? From a point S in plane MN a line is drawn perpendicular to plane MN intersecting PQ in R. ST is drawn perpendicular to AB. $TC = TH$. Prove that $\triangle CRS \cong \triangle HRS$.

3. In the figure, do RH, RT, and RC all lie in the same plane? Why? Using the hypothesis of Exercise 2, prove that $\triangle RHT \cong \triangle RTC$. Prove that $RT \perp HC$.



(Exs. 2–3)

4. The figure at the right represents a cube. AHC represents a plane passed through A, H, and C. If HB is 8 inches, find the length of AH, AC, and HC. How many degrees are there in angle HAC?



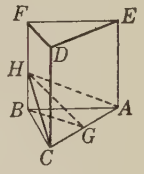
(Ex. 4)

5. Triangle ABC is an equilateral triangle drawn in a horizontal plane MN. The altitudes AD and BE intersect at S. SR is perpendicular to plane MN. AB is 12 inches, and SR is 4 inches. Find the length of the lines drawn from R to the vertices of triangle ABC. Draw triangles ABC and BRS in their correct proportions. (See § 26.)

6. Circle O is a circle in a vertical plane MN. AC is a diameter. From B, a point on circle O, BD is drawn perpendicular to AC, meeting it at D. BH is perpendicular to the plane MN. Lines are drawn from H to A, C, and D. Angle BAC is 30°, AC is 10 inches, and BH is 5 inches.

Evaluate all the lines of the figure. How many degrees are there in angle BCH? (See § 26.)

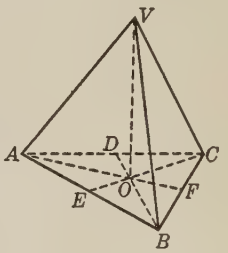
7. Figure $ABCDEF$ represents a right triangular prism. HAC represents a plane determined by AC and a point H on BF. Angle ABC is a right angle. EA, FB, and DC are perpendicular to plane ABC. BG and HG are perpendicular to AC (see Exercise 3). If AC is 15 inches, BC is 8 inches, and HB is 6 inches, find the length of AH, HC, BG, and HG.



(Ex. 7)

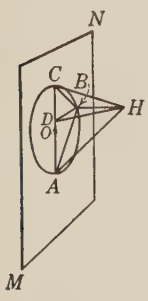
8. V–ABC is a pyramid whose base ABC is an equilateral triangle. O is the point of intersection of the medians AF, BD, and CE of triangle ABC, and VO is perpendicular to the plane ABC. Prove that $\triangle VOB \cong \triangle VOC \cong \triangle VOA$.

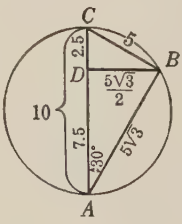
9. In the figure for Exercise 8, prove that VE is perpendicular to AB.

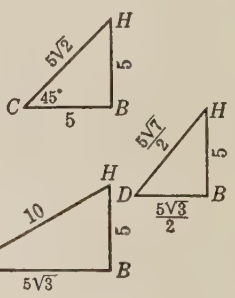


(Ex. 8)

§ 26. **Procedure with complicated figures.** When figures are at all complicated, it is helpful to redraw the important parts of the figure in their correct proportions as if they were in the plane of the paper. To illustrate this idea, the figures for Exercise 6, page 23, have been drawn in this fashion and the parts of the figure have been marked with their given and computed values, as shown below. Draw the important parts of the figure for Exercise 5, page 23, in the same way.





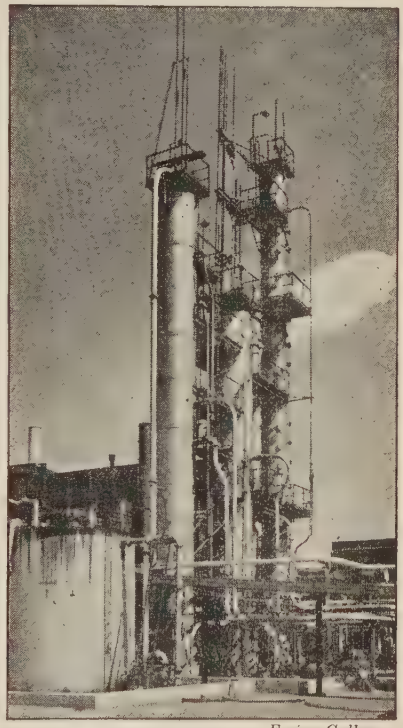


CHAPTER II

PERPENDICULAR LINES AND PLANES

Solids and Their Component Parts

There are numerous examples of geometric solids in the world around us — cubes, prisms, cylinders, spheres, and many others. Some of these solids are bounded entirely by plane surfaces; others have a combination of plane surfaces and curved surfaces; one solid is completely bounded by one curved surface. Which solids come under each of these classifications?



Ewing Galloway

Note the parallel planes and perpendicular lines.

Note that the *surfaces* meet in *lines*, straight or curved, and that the *lines* meet in *points*. These parts of the solids bear various relationships to each other. Lines may be oblique, perpendicular, or parallel to lines or planes. Planes may be oblique, perpendicular, or parallel to each other. When planes meet they form an angle analogous to the angle between two lines. In your study of this and the next three chapters, you will learn of the relationships among lines and planes in space so that you will be able to see and understand the relationships among the component parts of the solids you will study later.

Perpendicular Lines and Planes

(1) In space how many lines can be drawn perpendicular to a given line at a given point on the line? Can you deduce your answer from previous postulates and corollaries?

(2) At the middle point B of line AC, BE and BD are drawn perpendicular to AC. Plane MN is passed through BE and BD. (Give the authority for this construction.) R and S are points on BD and BE, respectively. (What relation have BD and BE to AC?) Prove that $\triangle ARS \cong \triangle RCS$.

(3) In Exercise (2) AB is apparently perpendicular to MN. Have you any method of proving that a line is perpendicular to a plane? Is this method based upon a definition, a postulate, or a theorem?

(4) Study the statement of Theorem 1, page 27, and draw a figure for it. Without referring to the printed hypothesis and conclusion, state them in good mathematical language. Check your statement with the printed one.

(5) Read the first paragraph under the heading "Proof" in Theorem 1 and then tell what must be done in order to prove that AB is perpendicular to MN.

For Exercises 6, 7, and 8 refer to the figure and hypothesis of Theorem 1.

(6) Name three pairs of apparently congruent triangles which have AH and CH as corresponding sides. Can any one of these pairs be proved congruent?

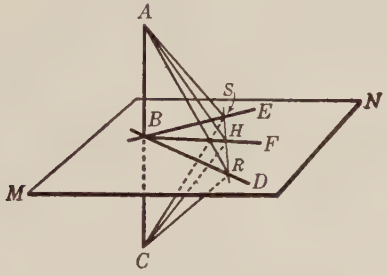
(7) List all the pairs of apparently congruent triangles that appear in the figure. Prove as many as possible of these pairs of triangles congruent. (All the pairs of triangles may be proved congruent if they are considered in the correct order.)

(8) Prove $AR = RC$ without proving $\triangle ABR \cong \triangle BRC$.

(9) Review the plane geometry methods of proving one line perpendicular to another. (See pages 232–233.)

THEOREM 1

§ 27. If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane of the two lines.



Given $AB \perp BE$ and BD at B; MN is the plane of BE and BD.

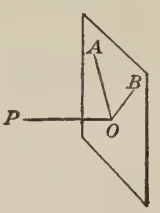
To prove that $AB \perp MN$.

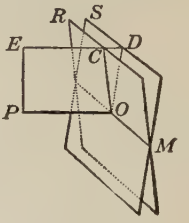
Proof. Through B draw in MN any straight line BF other than BE or BD. Extend AB to C, making $BC = AB$. Draw RS intersecting BD, BF, and BE at R, H, and S, respectively. Connect R, H, and S with A and C.

STATEMENTS	REASONS
1. $AB \perp BD$ and BE.	1. Given.
2. $AB = BC$.	2. Construction.
3. $AR = RC$, $AS = SC$.	3. Syllabus 110.[1]
4. $RS = RS$.	4. Identity.
5. $\triangle ARS \cong \triangle CRS$.	5. Why?
6. $\angle ARS = \angle CRS$.	6. Why?
7. $RH = RH$.	7. Identity.
8. $\triangle ARH \cong \triangle CRH$.	8. Why?
9. $AH = HC$.	9. Why?
10. $BF \perp AC$ at B or $AB \perp BF$.	10. Syllabus 30.
11. AB is perpendicular to every line through its foot in plane MN.	11. BF represents any line in plane MN through B.
12. $AB \perp MN$.	12. § 23.

[1] A syllabus of propositions in plane geometry is given on pages 231–242.

§ 28. COROLLARY. **One and only one plane can be passed perpendicular to a given line at a given point on the line.**

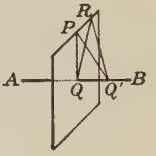




(1) Let O be the given point on the given line PO. Draw OA and OB each perpendicular to PO at O. The plane of OA and OB is perpendicular to PO at O (§ 27).

(2) Assume that planes MR and MS are each perpendicular to PO at O. Pass plane EO through PO, as shown. CO and DO would both be perpendicular to PO at O by § 23, but this is impossible by Syllabus 4.

§ 29. COROLLARY. **Through a given point not on a given line, one and only one plane can be passed perpendicular to the line.**



(1) Let AB be the given line and P be the given point. Draw PQ perpendicular to AB and from Q draw QR, some other perpendicular to AB. Then PQ and QR determine a plane perpendicular to AB at Q (§ 27). Hence one perpendicular plane can be drawn to AB from P.

(2) Assume that another plane can be drawn through P perpendicular to AB. Let this plane intersect AB at Q'. Draw PQ'. PQ and PQ' would then both be perpendicular to AB (§ 24), which is impossible (Syllabus 5).

EXERCISES

1. What relation has the intersection of two walls of your classroom to the plane of the floor?

2. Represent a vertical line AB and two lines BC and BD perpendicular to AB at B and also perpendicular to each other. Prove that any one of them is perpendicular to the plane determined by the other two.

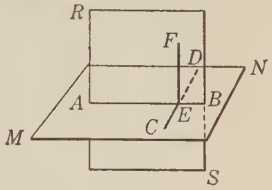
3. Show that each edge of a cube is perpendicular to the planes of two faces.

4. How can a carpenter, by using his square, test whether or not an upright is perpendicular to the floor?

5. CD is a line in plane MN. Can you construct a plane perpendicular to CD at a point E on it? Can you construct a line FE perpendicular to the plane MN?

6. Represent a line CD in a horizontal plane MN. At any point E of CD draw a plane RS perpendicular to CD. Through E draw any line HK in the plane RS. What is the relation of CD to HK?

7. CD is any line in plane MN. Plane RS is perpendicular to CD at E. FE, in the plane RS, is perpendicular to AB, the intersection of the two planes. Prove that FE is perpendicular to the plane MN. (What method of showing that a line is perpendicular to a plane can be used?)



(Ex. 7)

8. Explain a method of constructing a line perpendicular to a given plane at a given point on the plane.

Preparation for Theorem 2. The following exercises will help to prepare you for the next theorem.

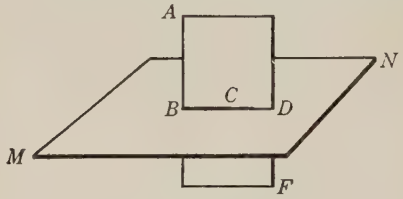
(1) Represent a vertical line with five lines drawn perpendicular to it at a given point. Make a model of these lines. Will it stand flat on the table?

(2) Study the statement of Theorem 2. What is the relation of MN to AB in the hypothesis? What is the relation of BC to AB in the hypothesis? What is the relation of BC to MN in the hypothesis? What is to be proved?

(3) Study the plan of Theorem 2. In what planes is BD? What is the relation of BD to AB? What is the relation of BC to AB? What is the relation of BD to BC?

THEOREM 2

§ 30. If a plane is perpendicular to a given straight line at a given point, it contains all straight lines perpendicular to the given line at the given point.



Given AB perpendicular to plane MN at B and any line BC perpendicular to AB at B.

To prove that BC lies in plane MN.

Plan. A plane AF through AB and BC will intersect MN in some line such as BD. We can show that BD and BC are the same line.

Proof.

STATEMENTS	REASONS
1. Pass a plane AF through AB and BC.	1. Is this possible?
2. AF intersects MN in BD.	2. Why?

The rest of the proof is left for you.

§ 31. Theorem 2 may also be stated: **All the perpendiculars that can be drawn to a straight line at a given point lie in a plane which is perpendicular to the line at the given point.**

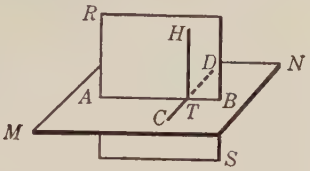
EXERCISES

1. Plane MN is perpendicular to a line AB at B. RS is a line in plane MN through B. CB is in plane MN and perpendicular to RS. Make a drawing and show that any line perpendicular to RS at B lies in the plane determined by AB and BC.

2. Points A, B, and C, which do not lie in a straight line, are each equidistant from the ends of a line segment EF. Prove that they all lie in a plane perpendicular to EF at its middle point.

THEOREM 3

§ 32. At a given point in a plane there can be one line perpendicular to the plane.



Given any point T in plane MN.

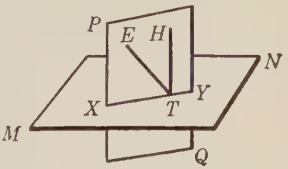
To prove that through point T one line can be drawn perpendicular to MN.

Proof. Draw line CD in plane MN through T. Pass through T a plane RS perpendicular to CD (§ 28). The plane RS will intersect the plane MN in line AB (§ 19). In plane RS draw HT perpendicular to AB (Syllabus 4).

The rest of the proof is left for you.

§ 33. COROLLARY. **At a given point in a plane there can be only one perpendicular to the plane.**[1]

Let HT be one perpendicular to plane MN at T (§ 32). Draw any other line ET through T. Draw the plane PQ determined by HT and ET and let it intersect MN in XY. Can ET be perpendicular to XY?



EXERCISE

1. Let AB and CD be two lines intersecting at E. At E construct two planes perpendicular to AB and CD, respectively. Prove that the intersection of these two planes is perpendicular to the plane of AB and CD.

[1] Theorem 3 and the corollary are usually given together as one proposition. They have been separated here in order to emphasize the distinctness of the proofs of the two parts. The statements of the propositions should be closely associated.

Review Exercises

1. How many planes are determined by four lines passing through a given point, if no three of the lines lie in the same plane?

2. If one edge of a cube is 10 in., how long is a diagonal?

3. If one edge of a cube is a units, how long is a diagonal?

4. The line AB, which is 7 inches long, is perpendicular to plane MN at B. From A an oblique line AC 11 inches long is drawn to meet MN at C. BC is drawn and then rotated about AB. Find to the nearest tenth of a square inch the area of the circle described by the point C.

5. Prove: If a plane is perpendicular to a line at the middle point of the line, any point of the plane is equally distant from the extremities of the line.

6. At the point D, where the altitudes of an equilateral triangle ABC meet, a line DE 8 inches long is drawn perpendicular to the plane of the triangle. If the sides of the equilateral triangle are each 6 inches long, what is the area of triangle AEC?

7. Prove: If three points, not in the same straight line, are each equally distant from the extremities of a line, they determine a plane which is perpendicular to the given line at its middle point.

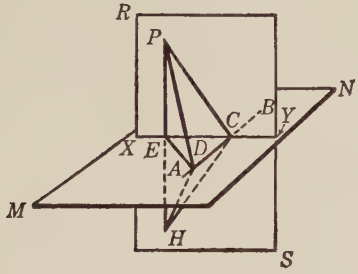
Preparation for Theorem 4. The following exercises will help you understand the proof of the next theorem.

(1) Represent a line KH in a plane MN and a point A not in the plane. Pass a plane PQ through A and perpendicular to KH intersecting MN in FG and KH at C. Give the authority for this construction. Draw AB in the plane PQ and perpendicular to FG at B. Continue AB its own length to E. Prove that $AC = EC$.

(2) In the figure for Exercise 1 take any point D on KH. Connect D with A, B, and E. Prove that triangle ACD and triangle ECD are congruent right triangles. What is the relation of BD to AE?

THEOREM 4

§ 34. Through a given external point there can be one line perpendicular to a given plane.



Given a point P not in the plane MN.

To prove that there can be one perpendicular to plane MN from point P.

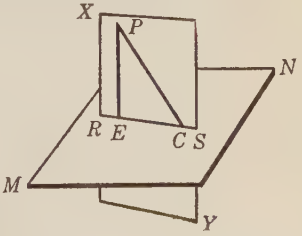
Plan. In order to show that $PD = DH$, prove that $\triangle PCD$ and $\triangle HDC$ are congruent right triangles. Then DE is the perpendicular bisector of PH. Use § 27.

Proof. In plane MN draw any line AB. Through P pass a plane RS perpendicular to AB (§ 29) and intersecting AB at C (§ 6) and MN in XY (§ 19). In plane RS draw PE perpendicular to XY. In plane MN draw any line ED other than XY through E and intersecting AB at D. Extend PE its own length to H. Draw PC, CH, PD, and DH.

The rest of the proof is left for you.

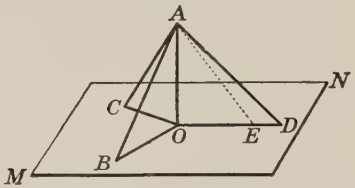
§ 35. COROLLARY. **Through a given external point there can be only one line perpendicular to a given plane.**

Let PE be one perpendicular from P to MN (§ 34) and PC any other line from P to MN. Draw the plane XY determined by PE and PC. XY will intersect MN in RS. PC cannot be perpendicular to RS.



Theorem 5

§ 36. Oblique lines drawn from a point in a perpendicular to a plane meeting the plane at equal distances from the foot of the perpendicular are equal; and, of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.



Given $AO \perp$ plane MN at O, and AB and AC any two oblique lines drawn from A to plane MN, making BO equal to CO, and AD drawn from A to MN, making $DO > CO$.

Part 1. To prove that $AB = AC$.

Proof. In $\triangle AOB$ and $\triangle AOC$,

1. $\angle AOB = \angle AOC =$ rt. $\angle$.	1. Why?
2. $AO = AO$, and $OB = OC$.	2. Why?
3. Hence $\triangle AOB \cong \triangle OAC$.	3. Syllabus 17.
4. $AB = AC$.	4. *c.p.c.t.e.*[1]

Part 2. To prove that $AD > AC$.

Proof. On OD lay off $OE = OC$; draw AE.

1. $AE = AC$.	1. Part 1.
2. $\angle AED$ is an obtuse angle.	2. Why?
3. Hence $\angle AED > \angle ADE$.	3. Why?
4. Hence $AD > AE$.	4. Why?
5. $AD > AC$.	5. Why?

(1) The *converse* of a theorem is obtained by interchanging its hypothesis and conclusion. State the converse of Theorem 5.

[1] Corresponding parts of congruent triangles are equal.

§ 37. COROLLARY. **Equal oblique lines drawn from a point in a perpendicular to a plane meet the plane at points equidistant from the foot of the perpendicular, and of two oblique lines, the greater meets the plane at the greater distance from the foot of the perpendicular.**

§ 38. COROLLARY. **The perpendicular is the shortest line from a point to a plane.**

The *distance* of a point from a plane is the shortest line and therefore the perpendicular from the point to the plane.

EXERCISES

1. Can a line be perpendicular to more than two lines in a plane if it does not lie in the plane?

2. If a circle is drawn in a plane and a perpendicular to this plane is drawn at the center of the circle, is a point on this perpendicular the same distance from every point on the circle?

3. AB is perpendicular to a plane MN at B; EF lies in the plane but does not contain B; BD is perpendicular to EF at D; $ED = DF$. Prove that $AF = AE$.

4. If, in Exercise 3, $BD = 4$ units, $DE = 3$ units, and $AB = 5$ units, how long are BE, BF, AE, AF, and AD?

5. AB is perpendicular to plane MN at B; AE and AF are equal oblique lines meeting MN at E and F. D is the middle point of EF. Prove that BD is perpendicular to EF.

6. If, in Exercise 5, $AF = 6$ units, $AD = 4$ units, and $BF = 5$ units, how long are DF, DE, BD, and AB?

7. AB and CD are perpendicular to plane MN at B and D, respectively; FE is in the plane MN and perpendicular to BD at D; F and E are taken on opposite sides of D so that $FD = DE$. Prove that CD, AD, and BD are perpendicular to FE.

8. V–$ABCD$ is a pyramid whose base $ABCD$ is square and whose other faces are equilateral triangles. Prove that a line from V to the intersection O of the diagonals AC and BD is perpendicular to the plane $ABCD$.

Review of Indirect Proof

The corollaries on page 28 were proved by *indirect proof*. Several propositions in the next chapter will be proved by the same method. It consists of showing that a statement is true by proving that the contradictory statement is false. A statement and its contradictory cannot both be true or false at the same time. If one is true, the other is false, and vice versa.

To prove that a statement is true by indirect proof, proceed as follows:

1. *Assume that the contradictory statement is true.*

2. *Draw conclusions from this contradictory statement until you reach a statement that is contrary to the hypothesis or some accepted proposition.*

Then you will know that the contradictory statement is false and that the statement itself is true.

Chapter Test

1. State a method of proving that a line is perpendicular to a plane.

2. Which is simpler: (1) to construct a plane perpendicular to a line, or (2) to construct a line perpendicular to a plane? Explain.

3. Complete: If a line is perpendicular to another line, it lies in a plane ___?___ to that ___?___.

4. P is a point not on AB. PQ is perpendicular to AB at Q. QR is another line perpendicular to AB at Q. Complete: PQ and QR ___?___ a plane ___?___ to ___?___ at ___?___.

5. Prove: If equal oblique lines are drawn from a given external point to a plane, they make equal angles with lines drawn from the points where the oblique lines meet the plane to the foot of the perpendicular from the point.

6. AB is perpendicular to the plane MN at B. From A a line is drawn oblique to the plane and meeting it at C. In the plane MN, CD and CE make equal angles with CB. Prove that $\angle ACD = \angle ACE$.

Maintaining Skills[1]

(*Computation*)

In computation you should make use of your knowledge of algebra wherever it may be helpful. Factor expressions when this will make the computation simpler. When carrying through a computation involving π or irrational numbers like $\sqrt{3}$, keep the numbers in this form until the end of the computation. Then, if you are required to give an approximate answer, substitute the approximate values.

1. Write as the product of a rational number and an irrational number:

$$\sqrt{12} \qquad \sqrt{45} \qquad \sqrt{27} \qquad \sqrt{20} \qquad \sqrt{72}$$

2. Express as a fraction with irrational numerators and rational denominators:

$$\sqrt{\tfrac{5}{4}} \qquad \sqrt{\tfrac{3}{8}} \qquad \sqrt{169 - \tfrac{5^2}{4}} \qquad \sqrt{2\tfrac{3}{4}}$$

3. Square the following numbers:

$$2\sqrt{3} \qquad \tfrac{1}{3}\sqrt{6} \qquad \frac{\sqrt{50}}{2} \qquad \sqrt{35} \qquad \frac{4\sqrt{3}}{3}$$

4. Write as a single irrational number or as the product of a rational number and an irrational number:

$$\sqrt{2} \times \sqrt{3} \qquad 2\sqrt{3} \times 5\sqrt{2} \qquad 4\sqrt{5} \times 6\sqrt{15}$$

5. Compute:

$$\tfrac{1}{3} \times 3.14 \times \left(\frac{25}{3.14 \times 4}\right)^2 \times 3 \div (\tfrac{5}{4} \times 1728)$$

6. Find the value of $2\pi rh + 2\pi r^2$ when $r = 5$ and $h = 8$.

7. Given $a = 6$. Find b when $b^2 = a^2 - \left(\frac{a}{2}\right)^2$.

   Then find $2A$ when $A = \frac{1}{2}ab$.
   Then find L which equals $3ab$.
   Finally find T which equals $2A + L$.

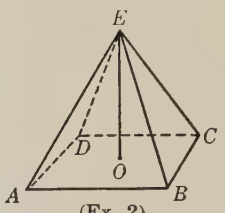
8. Given $a = 8$. Find b, when $b^2 = a^2 - \left(\frac{a}{2}\right)^2$.

   Find A which equals $\frac{1}{2}ab$.
   Find h, when $h = b^2 - (\frac{1}{3}b)^2$.
   Find V which equals $\frac{1}{3}Ah$.

[1] To the Teacher. See footnote on page 9.

Miscellaneous Exercises

1. At the center of a regular hexagon whose sides are each 6 inches a line 8 inches long is drawn perpendicular to the plane of the hexagon. The outer end of this perpendicular is connected with the vertices of the hexagon. Represent the figure. What is the total area of the six triangles thus formed?



(Ex. 2)

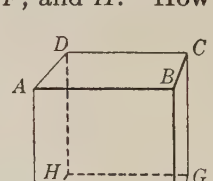
2. A certain pyramid has a square base, and the edges AB, AE, BE, CE, and DE are equal. Prove that a line connecting the center of the square with E is perpendicular to the plane of the square.

3. If two equal lines are drawn from an external point to a plane, they make equal angles with the perpendicular from that point to the plane.

4. Prove the converse of Exercise 3.

5. Can six lines of equal length be placed so as to form four congruent equilateral triangles? Make a model for this exercise, using toothpicks.

6. In a copy of the cube represented in the figure indicate the planes determined by the points A, C, F, and H. How many planes are determined? Show that four congruent equilateral triangles are formed by the intersection of these planes with the faces of the cube.



(Ex. 6)

7. A pyramid V–ABC is formed by using four equilateral triangles. (ABC is the base of the pyramid and V the vertex.) Prove that a line from any vertex perpendicular to the opposite face meets it at the center of the equilateral triangle.

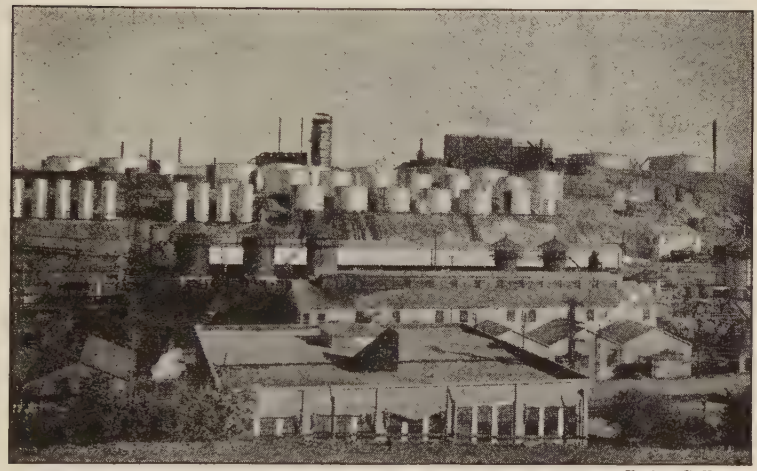
8. If, in Exercise 7, AB is 6 inches, how long is the line from V perpendicular to the plane ABC?

9. If, in Exercise 7, AB is a units, how long is the line from V perpendicular to plane ABC?

CHAPTER III

PARALLEL LINES AND PLANES

Parallel Planes in Common Objects

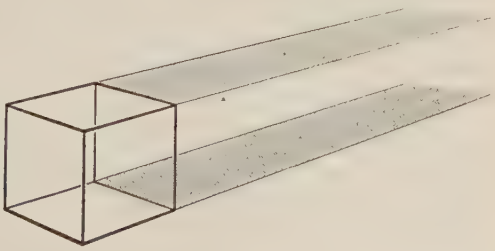


Ewing Galloway

An oil refinery at Martinez, California.

Suggestions of parallel planes are all about you; in the picture above for instance, the roof of the building in the foreground and the tops of the cylindrical tanks beyond lie in parallel planes. The cylindrical tin cans in the grocery store have two circular surfaces in parallel planes. The covers of a textbook, when it is new, are parallel.

It is interesting to note that parallel planes drawn in perspective converge to a straight line, just as parallel lines converge to a point. The diagram shows the convergence of two parallel surfaces of a cube.



Parallel Lines and Planes

§ **39.** If a line and a plane cannot meet however far they are extended, the line is parallel to the plane and the plane is parallel to the line. Likewise, if two planes cannot meet however far they are extended, the planes are parallel to each other. Conversely, if a line and a plane are parallel or if two planes are parallel, they cannot meet however far extended. Also, if a line and a plane meet or if two planes meet, they are not parallel.

§ **40.** *Skew lines* are lines which are not in the same plane and will never meet, however far they may be produced.

(1) Indicate parallel lines in the classroom.

(2) Indicate parallel planes in the classroom.

(3) Hold a yardstick so that it will represent a line parallel to the plane of the blackboard.

(4) Hold two yardsticks in such a way as to represent skew lines.

(5) Indicate skew lines in the classroom.

Preparation for Theorem 6. For the following exercises refer to the figure for Theorem 6, on page 41.

(1) AB is perpendicular to plane MN at B; BD is perpendicular at D to line EF in plane MN; $FD = DE$; AE, AF, BE, and BF are drawn. Prove that $\triangle AFE$ is isosceles.

(2) Prove that AD is perpendicular to EF.

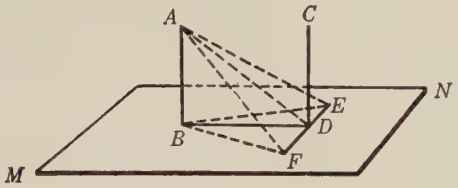
(3) At D in the figure draw DC perpendicular to plane MN. Prove that DC, DA, and DB all lie in one plane. Does AB lie in the same plane?

(4) Have you any authority for saying that two lines perpendicular to the same plane are parallel?

(5) In order to establish the fact that two lines are parallel, is it sufficient to prove that they are perpendicular to the same line?

THEOREM 6

§ 41. Two lines perpendicular to the same plane are parallel.



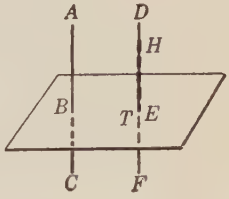
Given AB and CD perpendicular to MN at B and D, respectively.

To prove that $AB \parallel CD$.

Plan. Prove that CD, AD, and BD are perpendicular to EF at D. CD, AD, and BD lie in the same plane (§ 31). AB is in this plane. Why? AB and $CD \perp BD$ and lie in the same plane. $AB \parallel CD$. Why?

Proof. Draw BD. Draw EF in MN perpendicular to BD at D. Make $DE = DF$. Draw BE, BF, AE, AF, and AD.

The rest of the proof is left for you.



§ 42. COROLLARY. **If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.**

Through any point in the second line construct a perpendicular to the plane and prove that the line coincides with the perpendicular.

§ 43. COROLLARY. **If two straight lines are parallel to a third straight line, they are parallel to each other.**

A plane perpendicular to the third line will be perpendicular to the other two lines.

§ 44. COROLLARY. **If from the foot of a perpendicular to a plane a straight line is drawn at right angles to any line in the plane, the line drawn from its intersection with the line in the plane to any point of the perpendicular is perpendicular to the line of the plane.**

Preparation for Theorems 7, 8 and 9. (1) What is meant by the intersection of two surfaces?

(2) Two planes MN and PQ intersect in the line AB.

(*a*) Can a line be drawn in the plane PQ in such a way that it will meet MN without meeting AB?

(*b*) If a line is drawn in the plane PQ and parallel to AB, will it meet MN? (If it meets MN, in what line must the intersection be?)

(*c*) Will a line drawn in PQ so that it intersects AB meet the plane MN? Why?

(3) If two lines are parallel, is every plane containing one of the lines, and only one, parallel to the other? (Pass a plane through the two parallel lines and follow the reasoning suggested in Exercise 2.)

(4) If a line is parallel to a plane, is it parallel to the intersection of this plane with any plane passed through the line?

(5) If two planes have a point in common, how many other points have they in common? Explain your answer.

(6) If two planes are parallel, can a line be drawn in one of them in such a way that it will intersect the other? (Indirect proof.)

(7) If two planes are parallel, will any line drawn in one be parallel to any line drawn in the other?

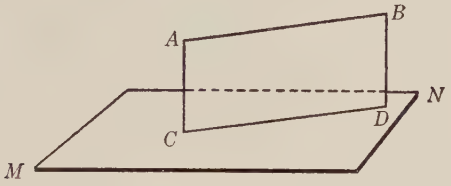
(8) How many planes can be drawn through one of two parallel lines and parallel to the other line?

(9) If two lines are parallel to the same plane, are they parallel to each other? Discuss the possibilities.

(10) If two parallel planes are cut by a third plane, are the lines of intersection parallel?

THEOREM 7

§ 45. If a line is parallel to a plane, it is parallel to the intersection of this plane with any plane passed through the line.



Given the line $AB \parallel$ plane MN, and CD, the intersection of plane MN and a plane through AB.

To prove that $AB \parallel CD$.

Plan. Show that the lines are in the same plane and cannot meet.

Proof.

STATEMENTS	REASONS
1. AB and CD lie in the plane $ABCD$.	1. Why?
2. If AB intersects CD, it intersects MN.	2. Why?
3. But $AB \parallel MN$.	3. Why?
4. Hence $AB \parallel CD$.	4. Why?

§ 46. COROLLARY. **If a line is parallel to two intersecting planes, it is parallel to their intersection.**

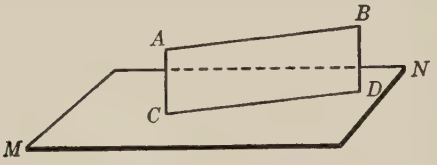
Suppose it were not. Pass a plane through the line and any point of the intersection. Then by § 45 we should have two lines parallel to the same line through the same point. This is impossible.

§ 47. COROLLARY. **If two intersecting lines are each parallel to a plane, the plane of these lines is parallel to that plane.**

If the two planes should meet, the line of intersection would be parallel to each of the given lines (§ 45), which is impossible.

THEOREM 8

§ 48. If two lines are parallel, every plane containing one of the lines, and only one, is parallel to the other.



Given $AB \parallel CD$ and MN containing CD, but not AB.

To prove that $AB \parallel MN$.

Plan. AB and CD determine a plane intersecting MN in CD. If AB meets plane MN, it must meet it in CD. Why? But this is impossible, for $AB \parallel CD$. Hence plane $MN \parallel AB$.

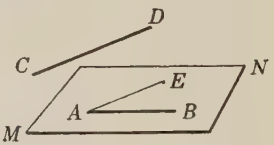
Proof. The proof is left for you.

§ 49. COROLLARY. **If two planes intersect, a line drawn in one, parallel to the intersection, is parallel to the other.**

§ 50. COROLLARY. **If two intersecting lines are respectively parallel to two other intersecting lines but do not lie in the same plane with them, the planes determined by the two pairs of intersecting lines are parallel.** (Use §§ 48, 47.)

§ 51. COROLLARY. **Through either of two skew lines one plane, and only one, can be passed parallel to the other.**

Through any point of AB draw a line parallel to CD and draw the plane determined by the intersecting lines. (Use § 48.) If another plane could be passed through AB parallel to CD, AB would be parallel to CD (§ 46).

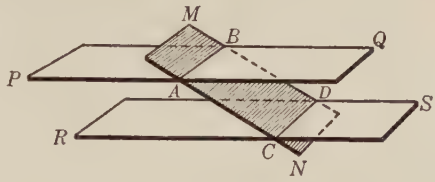


§ 52. COROLLARY. **Through any given point one plane, and only one, can be passed parallel to each of two skew lines.**

Through the point draw a line parallel to each of the given lines. This construction will give one plane parallel to the two skew lines. Could two planes be passed through the point and parallel to the two skew lines?

THEOREM 9

§ 53. If two parallel planes are cut by a third plane, the lines of intersection are parallel.



Given parallel planes PQ and RS, cut by plane MN, and AB and CD, their respective intersections with MN.

To prove that $AB \parallel CD$.

Plan. The two lines of intersection are in the same plane. Show that they cannot meet.

Proof. The proof is left for you.

§ 54. COROLLARY. **Parallel line segments included between parallel planes are equal.**

§ 55. COROLLARY. **The perpendicular distance between two parallel planes is everywhere the same.**

EXERCISES

1. How would you construct a line parallel to a given plane through a point outside the plane?

2. How would you construct a plane parallel to a given plane through a point not in the plane?

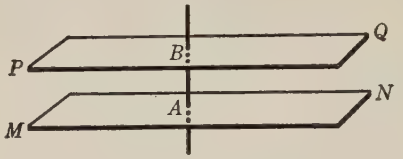
3. How would you construct a line through a given point and parallel to two given intersecting planes? Prove your construction.

4. How would you construct a line through a given point and parallel to two parallel planes?

5. MN is a given plane. AB and CD are parallel lines not in MN and not parallel to MN. AE, BF, CG, and DH are perpendicular to MN at E, F, G, and H respectively. Prove that EF is parallel to GH.

Theorem 10

§ 56. Two planes perpendicular to the same line are parallel.



Given planes MN and PQ perpendicular to AB.

To prove that $MN \parallel PQ$.

Proof. MN and PQ are either parallel or not parallel. If they are not parallel, they will meet at some point K. There will then be two planes perpendicular to AB from K. This is impossible. The only remaining possibility is that they are parallel.

EXERCISES

Prove the following exercises:

1. The plane of any face of a cube is parallel to the plane of the opposite face.

2. If two planes intersect, no line can be constructed perpendicular to both of them. (Indirect proof.)

3. Two lines perpendicular, respectively, to two intersecting planes cannot be parallel.

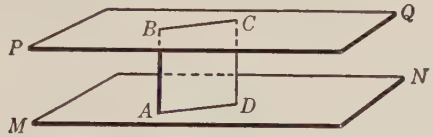
Preparation for Theorem 11. Use the figure on page 47 for the following exercises.

(1) MN and PQ are parallel planes. AB is perpendicular to PQ at B and intersects MN at A. By passing a plane through AB show that AB is perpendicular to any line in MN through its foot.

(2) Study the statement and the first step in the proof of Theorem 11. What must be proved to establish the fact that AB is perpendicular to MN?

Theorem 11

§ 57. A line perpendicular to one of two parallel planes is perpendicular to the other also.



Given plane $MN \parallel$ plane PQ; $AB \perp PQ$.

To prove that $AB \perp MN$.

Proof.

Statements	Reasons
1. In MN draw any line AD and pass a plane through AB and AD.	1. § 13.
2. This plane will intersect PQ in BC.	2. Why?
3. $AB \perp PQ$ and $MN \parallel PQ$.	3. Given.
4. $AB \perp BC$.	4. Why?
5. $AD \parallel BC$.	5. Why?
6. $AB \perp AD$.	6. Why?
7. AB is perpendicular to every line in MN through its foot.	7. AD represents any line in plane MN through A.
8. $AB \perp MN$.	8. Why?

§ 58. Corollary. Through a given point outside a plane one plane, and only one, can be passed parallel to the given plane.

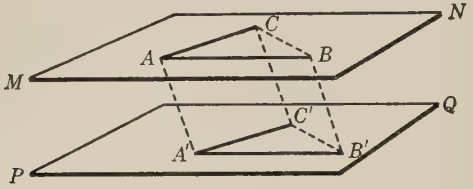
How may one plane be drawn parallel to a given plane through a given point? Suppose that another plane is drawn parallel to the plane through the point.

§ 59. Corollary. Two planes perpendicular to two intersecting lines are not parallel. (Indirect proof.)

(1) $\triangle ABC$ and $\triangle A'B'C'$ are in parallel planes. AA', BB', and CC' are parallel to each other. Prove that $ABB'A'$, $BCC'B'$, and $ACC'A'$ are parallelograms.

THEOREM 12

§ 60. If two angles not in the same plane have their sides respectively parallel and lying on the same side of the straight line joining their vertices, they are equal and their planes are parallel.



Given the sides of $\angle BAC$ and $\angle B'A'C'$ respectively parallel and lying on the same side of AA'. MN is determined by AB and AC. PQ is determined by $A'B'$ and $A'C'$.

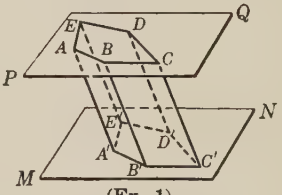
To prove that $\angle BAC = \angle B'A'C'$ and $MN \parallel PQ$.

Plan. Prove that $ABB'A'$, $ACC'A'$, and $BCC'B'$ are parallelograms and that $\triangle BAC \cong \triangle B'A'C'$. $MN \parallel PQ$ (§ 50).

Proof. Make $AB = A'B'$ and $AC = A'C'$. Draw AA', BB', CC', BC, and $B'C'$.

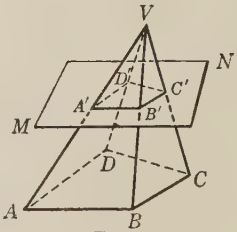
EXERCISES

1. AA', BB', CC', DD', and EE' are parallel lines cutting PQ in A, B, C, D, E, and MN in A', B', C', D', E', respectively. $PQ \parallel MN$. Prove that polygon $ABCDE \cong$ polygon $A'B'C'D'E'$.



(Ex. 1)

2. V–$ABCD$ is a pyramid. Plane MN is parallel to the base of the pyramid, cutting the faces of the pyramid in $A'B'C'D'$. Prove that $A'B'C'D' \sim ABCD$.

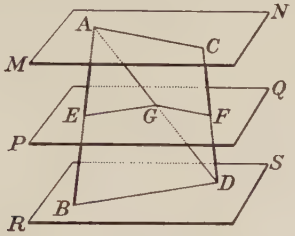


(Ex. 2)

3. Under what conditions are two angles, not in the same plane, with their sides parallel respectively, supplementary instead of equal, as in Theorem 12?

THEOREM 13

§ 61. If two lines are cut by three parallel planes, the corresponding segments are proportional.



Given AB and CD, two lines cut by parallel planes MN, PQ, and RS at A, E, B, and C, F, D, respectively.

To prove that $\frac{AE}{EB} = \frac{CF}{FD}$.

The plan is to form two triangles ABD and ADC with a common side AD, each with a line parallel to the base, and form two proportions of which one ratio will be common.

Proof. Draw AD intersecting PQ at G. Through AB and AD pass a plane intersecting PQ in EG and RS in BD. Pass a plane through AD and CD intersecting PQ in GF and MN in AC.

STATEMENTS	REASONS
1. EG is $\parallel$ to BD.	1. § 53.
2. FG is $\parallel$ to AC.	2. Why?
3. Hence $\frac{AE}{EB} = \frac{AG}{GD}$.	3. Syllabus 124.
4. Also, $\frac{CF}{FD} = \frac{AG}{GD}$.	4. Why?
5. $\frac{AE}{EB} = \frac{CF}{FD}$.	5. Why?

§ 62. COROLLARY. **If three parallel planes cut off equal segments on one transversal, they cut off equal segments on any transversal.**

Of what theorems in plane geometry do the above theorem and corollary remind you?

EXERCISES

Draw a figure similar to the figure for Theorem 13, lettering it the same as the figure for the theorem. Solve the following exercises with reference to the figure you have drawn:

1. If $AE = 3$ inches, $EB = 5$ inches, and $CF = 4$ inches, how long is FD?

2. If $AE = a$ units, $EB = b$ units, and $CF = c$ units, how long is FD?

3. If $AB = 10$ inches, $AE = 4$ inches, and $CD = 15$ inches, how long are CF and FD?

4. If $AB = a$ units, $AE = b$ units, and $CD = c$ units, how long are CF and FD?

5. If $AE = 2$ inches, $EB = CF$, and $FD = 3$ inches, how long is CD?

6. If $EB = \frac{5}{2}\,AE$ and $CF = a$ units, how long is FD?

7. If $AE = a$ units, $EB = 2\,CF$, and $FD = b$ units, how long is AB?

8. If $AE = EB$ and $CF = 2\frac{3}{4}$ inches, how long is CD?

9. If $AB = 7\frac{3}{4}$ inches, $AE = 2\frac{7}{8}$ inches, and $CD = 14$ inches, how long are CF and FD?

10. If $AE = 5\frac{3}{16}$ inches, $EB = CF$, and $FD = 7\frac{3}{4}$ inches, how long is CD?

11. If $AE = 3.64$ units, $EB = 2\,CF$, and $FD = 7.56$ units, how long is AB?

12. If $AB = 5.2$ units, $CD = 10.4$ units, and $FD = 3\,AE$, how long are AE, EB, CF, and FD?

13. If $\frac{AE}{EB} = \frac{2}{3}$ and $CF = 2\frac{1}{3}$ inches, how long is CD?

14. If $EB = 3$ inches, $CD = 10$ inches, and $FD = 3\,AE$, how long are AE, CF, and FD?

15. If $FD = 2\frac{2}{3}$ inches, $AB = 18$ inches, and EB is $\frac{1}{5}\frac{2}{}\,CF$, how long are AE, EB, and CF?

16. If $EB = a$ units, $CD = b$ units, and $FD = 3\,AE$, how long are AE, CF, and FD?

Problem-Solving Helps

Can you complete the following statements concerning helpful methods of proving propositions?

1. To prove that a line and a plane are perpendicular, show that —
   - (*a*) the line is perpendicular to any line in the plane ___?___.
   - (*b*) the line is perpendicular to each of two ___?___.
   - (*c*) the line is parallel to another line that is ___?___.
   - (*d*) the line is perpendicular to a plane that is parallel to ___?___, or that the plane is parallel to a plane that is ___?___.
   - (*e*) the line coincides with a line that is known to be ___?___.

2. To prove that a line and a plane are parallel, show that —
   - (*a*) the line and the plane cannot ___?___ (using the indirect method).
   - (*b*) the line is parallel to a line in ___?___ (that the plane contains a line that is ___?___).
   - (*c*) that the line coincides with a ___?___.

3. To prove two lines are parallel, show that —
   - (*a*) they cannot ___?___.
   - (*b*) they are perpendicular to ___?___.
   - (*c*) they are the intersections of a plane with ___?___.
   - (*d*) one of them is the intersection of two planes, and the other lies in one of the planes and is ___?___.
   - (*e*) one of them is the intersection of two planes, and the other is parallel ___?___.

4. To prove that two planes are parallel, show that —
   - (*a*) they cannot ___?___.
   - (*b*) two intersecting lines in one plane are ___?___.
   - (*c*) two intersecting lines are parallel, respectively ___?___.
   - (*d*) they are perpendicular to ___?___.

Chapter Test

Complete the statements in Exs. 1–13.

1. If two intersecting lines are, respectively, parallel to two other intersecting lines · · ·.

2. If two lines are cut by three parallel planes, · · ·.

3. If one of two parallel lines is perpendicular to a plane, · · ·.

4. A line perpendicular to one of two parallel planes · · ·.

5. If two parallel planes are cut by a third plane, · · ·.

6. If two lines are parallel, every plane containing one · · ·.

7. If two lines are parallel to a third line, · · ·.

8. If two angles not in the same plane have their sides, respectively, parallel and · · ·.

9. If two planes intersect, a line drawn in one parallel to the intersection · · ·.

10. Parallel line segments included between · · ·.

11. If a line is parallel to a plane, it is parallel to the intersection of · · ·.

12. Two planes perpendicular to the same line · · ·.

13. If a line is parallel to two intersecting planes, · · ·.

14. If a line and a plane are perpendicular to the same line at different points, are they parallel?

15. Are two lines perpendicular to the same line parallel?

16. If two planes are parallel, is a line in one of them parallel to any line drawn in the other?

17. Can three lines, no two of which are parallel, be intersected by a fourth straight line?

18. Are two lines parallel to the same plane parallel to each other?

19. Are two lines perpendicular to the same plane parallel to each other?

20. Three lines, BD, AD, and CD, are each perpendicular to FE at D. Are BD, AD, and CD in the same plane? Is AB in this plane? Explain.

21. If two planes are parallel, can a line drawn in one of them meet the other? Explain.

Miscellaneous Exercises

1. Show that a plane may be passed through any two opposite edges of a cube or a rectangular solid. Can a plane be passed through any two edges of a cube or a rectangular solid?

2. If each edge of a cube is $2a$ inches, how long is a diagonal of the cube?

3. A pyramid is formed by using a square 6 inches on a side for a base and four isosceles triangles whose equal sides are each 8 inches. How long is the line from the point where the four triangles meet to the middle point of the base?

4. Prove: If a straight line is bisected by a plane, the extremities of the line are equally distant from the plane.

5. Prove: If two points are on opposite sides of a plane and equally distant from it, the line connecting these two points is bisected by the plane.

6. Prove: A line in one of two intersecting planes cannot be parallel to a line in the other unless both lines are parallel to the intersection.

7. In a plane quadrilateral it has been shown that lines connecting the middle points of the sides in succession form a parallelogram. Is this also true if the four vertices of the quadrilateral do not all lie in the same plane?

8. A rectangle is rotated around one of its sides. Show that the surfaces generated by the two sides adjacent to this side are in parallel planes. (The surface generated by the fourth side is a ***cylindrical*** surface.)

9. Two planes MN and PQ are parallel and 12 inches apart. A line AB 18 inches long moves so that A is always in MN and B is always in PQ. What is the locus of the middle point of AB?

10. Prove: A line which is parallel to a plane is at all points equally distant from the plane.

11. Prove: If a straight line and a plane are perpendicular to the same line, they are parallel.

12. Prove: If from any point of a perpendicular to a plane a straight line is drawn at right angles to any line in the plane, the line drawn from its intersection with the line in the plane to the foot of the given perpendicular is perpendicular to the line in the plane.

13. A plane is passed through one diagonal of a parallelogram. Prove that lines drawn perpendicular to it from the ends of the other diagonal are equal.

14. Prove: If a plane and a line are parallel, a line through any point of the plane and parallel to the line lies entirely in the plane.

15. Prove: If two points are on the same side of a plane and the same distance from the plane, the straight line through these points is parallel to the plane.

16. Prove: Three points not in the same straight line on the same side of a plane and equally distant from the plane determine a plane which is parallel to the given plane.

17. From two points on a line which is parallel to a plane parallel lines are drawn to the plane. Prove that the segments from the points to the plane are equal.

18. Prove: If two lines intersect, a line drawn parallel to one of them through any point of the other will lie in the plane of the intersecting lines.

19. Do three lines perpendicular to the same plane all lie in one plane?

20. Prove: If three lines are perpendicular to the same plane, the plane determined by any two of them is parallel to the third line (or contains it).

21. Prove: If a quadrilateral is formed by connecting four points which are not all in one plane, lines connecting the middle points of the opposite sides will bisect each other. (Note that it is necessary to show that these two lines lie in the same plane.)

CHAPTER IV

DIHEDRAL AND POLYHEDRAL ANGLES

Different Kinds of Angles

The term *angle* is itself such a basal term that it is not possible to give a satisfactory definition employing terms more simple than itself. It is one of our undefined terms. We must be content with *describing* it as a figure formed by two straight lines that meet.

We can, of course, define the *size* of an angle. An angle is measured by the amount of turning necessary to bring one of the lines into coincidence with the other. The early idea of an angle limited the concept to an angle less than 180°. Modern science demands angles of all sizes — a straight angle, an angle of 180°; a reflex angle, an angle between 180° and 360°; a *perigon*, an angle of 360°; and even angles exceeding 360°. For example, the minute hand of the clock turns through an angle of 540° in an hour and a half.

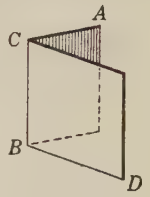
The original idea of angle has been extended to include the figure formed by a plane and a line meeting it (§ 116), by two intersecting planes (§ 63), by three planes having a common point (§ 89), and by any number of planes having a common point (§ 87). We speak also of the angle between two curves meeting at a point (§ 313) or even of the angle between two skew lines.

Ewing Galloway

The steeple on this church is in the form of a *polyhedral* angle.

Dihedral Angles

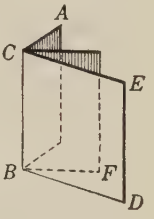
§ **63.** The figure formed by two planes which meet each other is called a ***dihedral angle.*** The intersection of the two planes is called the ***edge***, and the two planes are called the ***faces*** of the angle. Thus planes *AB* and *CD* meet in line *BC*, forming a dihedral angle. *BC* is the edge, and planes *AB* and *CD* are the faces of the angle.



(1) Illustrate a dihedral angle by opening a book. Designate the edge and the faces.

(2) May more than one dihedral angle have the same edge? Illustrate with a book.

§ **64.** When two planes meet, forming only one dihedral angle, it may be read by naming the edge, as dihedral angle *BC* in the above figure. When more than one angle is formed with the same edge, each must be designated by four letters as *A–BC–D*, *A–BC–F*, and *E–BC–F*, the two middle letters showing the edge and the other letters denoting points on the faces.



§ **65.** If two dihedral angles have a common edge and a common face separating the two angles, they are said to be ***adjacent.*** Dihedral angles *A–BC–F* and *E–BC–F* are adjacent. Are angles *A–BC–D* and *A–BC–F* adjacent?

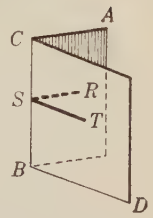
(3) Fold several pieces of paper to illustrate dihedral angles. Place these so that the edges of the angles coincide. Letter the angles and read them.

(4) Represent a cube and designate by four letters the dihedral angles formed. May the angles in this figure be designated by reading the edges only? How many dihedral angles are there?

(5) Point out dihedral angles in the classroom.

(6) Read the dihedral angles in the figures on pages 57–62.

Plane Angle of a Dihedral Angle



§ 66. An angle formed by two lines, one in each face of a dihedral angle and both perpendicular to the edge at the same point, is called a ***plane angle of the dihedral angle.*** Thus, $\angle RST$ is a plane angle of the dihedral angle A–BC–D if RS and ST are in planes AB and CD, respectively, and are perpendicular to BC at S. From this definition it is easy to deduce the following corollary:

§ 67. COROLLARY. **All plane angles of the same dihedral angle are equal.**

EXERCISES

1. Prove: If a plane is perpendicular to the edge of a dihedral angle, the angle formed by the intersections of this plane with the faces of the dihedral angle is a plane angle of the dihedral angle.

2. How many dihedral angles are there in a cube?

3. Prove: The edges of a cube form plane angles of the dihedral angles.

4. Prove: If a rectangular piece of paper is folded on a line perpendicular to one of its sides, that side will form a plane angle of the dihedral angle.

5. Prove: The plane determined by the sides of a plane angle of a dihedral angle is perpendicular to the edge of the dihedral angle.

6. Draw several lines through a point on a rectangular piece of paper. Fold the paper on a line which passes through this point and is perpendicular to a side of the paper. Open the paper to form a dihedral angle. Compare the various angles formed by the lines in the two planes with a plane angle of the dihedral angle. How small may an angle made in this way be? How large?

§ 68. Two dihedral angles are said to be ***congruent*** if they can be made to coincide.

THEOREM 14

§ 69. If two dihedral angles are congruent, their plane angles are equal.

The dihedral angles can be made to coincide. If the edge of one is moved along the edge of the other until the vertices of the plane angles coincide, the plane angles will coincide and are therefore equal.

THEOREM 15

§ 70. Two dihedral angles are congruent if their plane angles are equal.

If a plane angle of one is brought into coincidence with a plane angle of the other, the edges and faces of the two dihedral angles will coincide; hence they are congruent.

§ 71. The ***measure of a dihedral angle*** is the measure of a plane angle of the dihedral angle. Two dihedral angles are said to be equal if they have the same measure.

§ 72. COROLLARY. **Equal dihedral angles are congruent;** and, conversely, **congruent dihedral angles are equal.**

§ 73. Dihedral angles are ***complementary*** or ***supplementary*** if their plane angles are complementary or supplementary.

§ 74. Two dihedral angles are said to be ***vertical*** when the faces of one are the prolongations of the faces of the other.

§ 75. A dihedral angle is said to be ***right***, ***acute***, or ***obtuse***, according as its plane angle is right, acute, or obtuse.

§ 76. Two planes are ***perpendicular*** if they form a dihedral angle whose plane angle is a right angle.

(1) The following statements may be proved by constructing a plane perpendicular to the edge of the dihedral angles and showing the relation between the plane angles thus formed:

(*a*) When one plane meets another, the adjacent dihedral angles are supplementary.

(*b*) Two vertical dihedral angles are equal.

(*c*) If two parallel planes are cut by a third plane, the alternate interior dihedral angles are equal.

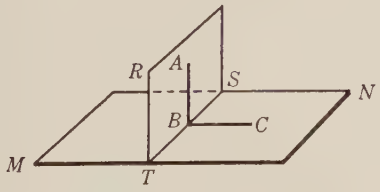
(2) Represent a line perpendicular to a plane and pass any plane through that line. Draw the plane angle of the dihedral angle formed.

(3) What means have you of proving that one plane is perpendicular to another?

(4) AB in the plane RS is perpendicular to the plane MN at B. BC in the plane MN is perpendicular at B to TS, the intersection of the two planes. Prove that $\angle ABC$ is the plane angle of the dihedral angle R–ST–N. Prove that RS is perpendicular to MN. (Refer to the figure for Theorem 16.)

THEOREM 16

§ 77. If a line is perpendicular to a given plane, every plane which contains this line is perpendicular to the given plane.



Given $AB \perp$ plane MN at B. RS is any plane containing AB and intersects MN in TS.

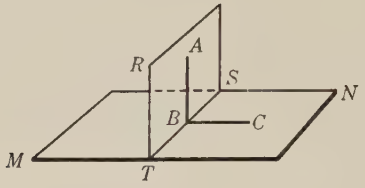
To prove that plane $RS \perp$ plane MN.

Plan. In plane MN draw $BC \perp TS$. Prove that $\angle ABC$ is the plane angle of the dihedral angle, and that it is a right angle.

Proof. The proof is left for you.

THEOREM 17

§ 78. If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.



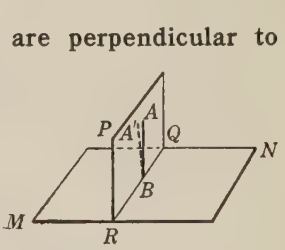
Given plane RS perpendicular to plane MN and intersecting it in TS. AB is in plane RS and is perpendicular to TS.

To prove that $AB \perp MN$.

Plan. In MN draw $BC \perp TS$. Prove that $\angle ABC$ is the plane angle of the dihedral angle and that it is a right angle.

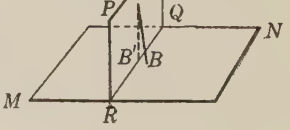
Proof. The proof is left for you.

§ 79. COROLLARY. If two planes are perpendicular to each other, a line perpendicular to one of them at a point of their intersection lies in the other.



If $A'B$ is drawn in the plane PQ and perpendicular at B to RQ, the intersection of the two planes, it will coincide with AB, the given line (§§ 78, 33).

§ 80. COROLLARY. If two planes are perpendicular to each other, a line perpendicular to one of them from any point of the other lies in the second plane.



If AB' is drawn in the plane PQ and perpendicular from A to RQ, the intersection of the two planes, it will coincide with AB, the given line (§§ 78, 35).

EXERCISES

1. Prove: If a plane is perpendicular to a line contained in another plane, the two planes are perpendicular.

2. Prove: A plane which is perpendicular to the intersection of two planes is perpendicular to both planes.

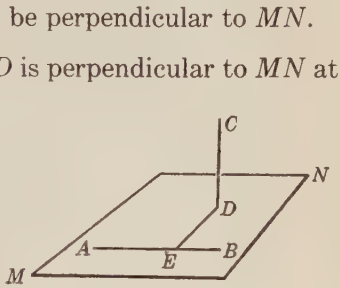
3. Prove: The plane determined by a plane angle of a dihedral angle is perpendicular to the faces of the dihedral angle.

4. How many planes may be passed perpendicular to a plane through a point in the plane? Give in detail the method of constructing one of these planes (§§ 32, 77).

5. How many planes may be passed perpendicular to a plane through a point outside the plane? Give in detail the method of constructing one of these planes (§§ 34, 77).

6. AB is oblique to plane MN. Tell how to construct a plane which will contain AB and be perpendicular to MN.

7. AB is in plane MN and CD is perpendicular to MN at D. $DE \perp AB$ at E. Prove that AB is perpendicular to the plane determined by CD and DE. (Note that DE is part of the intersection of the two planes.)



(Ex. 7)

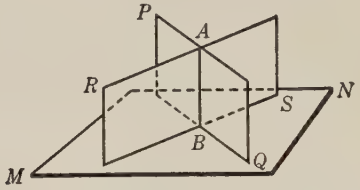
8. $ABCD$ is a square. EB is perpendicular to AB and BC. Prove that AC is perpendicular to the plane of EB and BD.

9. Prove: If each of two intersecting planes is perpendicular to a third plane, a line drawn from any point of the intersection of these two planes and perpendicular to the third plane lies in both the first two planes.

10. Plane RS is perpendicular to plane MN and intersects it in AS. CD is in RS and parallel to AS. BE is in MN and intersects AS at F. Prove that a line in RS and perpendicular to AS at F is perpendicular to BE and CD.

THEOREM 18

§ 81. If each of two intersecting planes is perpendicular to a third plane, their intersection is also perpendicular to that plane.



Given planes PQ and RS intersecting in AB and both perpendicular to MN.

To prove that $AB \perp MN$.

The plan is to draw $CD \perp MN$ from any point in AB, and show that CD coincides with AB.

Proof. From any point C in AB draw $CD \perp MN$.

STATEMENTS	REASONS
1. PQ and RS intersect in AB.	1. Given.
2. PQ and $RS \perp MN$.	2. Given.
3. C lies in both PQ and RS.	3. Why?
4. $CD \perp MN$.	4. Construction.
5. CD lies in both PQ and RS.	5. § 80.
6. CD is the intersection of PQ and RS.	6. Why?
7. CD and AB coincide.	7. Why?
8. $AB \perp MN$.	8. It coincides with CD, which was drawn $\perp MN$.

§ 82. COROLLARY. **Through a given line not perpendicular to a given plane one plane, and only one, can be passed perpendicular to the given plane.**

From any point of the line construct a perpendicular to the plane. Pass a plane through the given line and the perpendicular (§ 13). If more than one plane could be passed through the given line and perpendicular to the given plane, the line would be perpendicular to the given plane (§ 81).

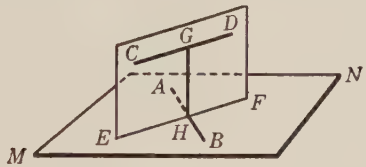
Perpendicular between Two Skew Lines

(1) Place a yardstick on a table and hold another in such a position that the two will represent skew lines. Find a position for a third yardstick such that it will be perpendicular to both lines.

(2) AB and CD are two skew lines. Through AB plane MN is passed parallel to CD. Through CD a plane is passed perpendicular to MN and intersecting MN in EF. Give the authority for these two constructions. Prove that EF intersects AB. (Indirect proof.)

THEOREM 19

§ 83. Between any two given skew lines there is one common perpendicular.



Given AB and CD, two skew lines.

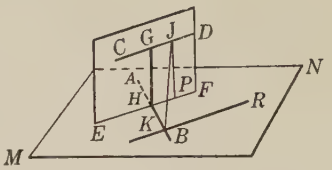
To prove that one common perpendicular can be drawn between AB and CD.

Proof.

STATEMENTS	REASONS
1. Pass plane MN through AB and parallel to CD.	1. § 51.
2. Pass plane CF through CD and perpendicular to MN, intersecting MN in EF.	2. § 82.
3. $CD \parallel EF$.	3. Why?
4. EF must intersect AB at some point H.	4. Why? (Suppose it did not.)
5. At H draw $HG \perp EF$ in plane CF.	5. Syllabus 4.

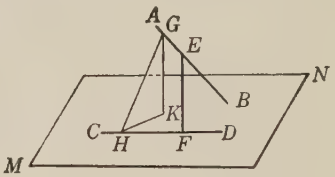
(The proof that HG is the required line is left for you.)

§ 84. COROLLARY. **Between any two skew lines only one common perpendicular can be drawn.**



HG is a common perpendicular to *AB* and *CD* (§ 83). Plane *MN* is parallel to *CD* and plane *CF* is perpendicular to plane *MN*. Let *JK* be any other possible perpendicular between *AB* and *CD*. Draw *JP* in plane *CF* and perpendicular to *EF*. A plane through *CD* and *JK* will intersect *MN* in *KR*, and *KR* will be parallel to *CD*. In this case *JK* will be perpendicular to *AB* and *KR* and hence perpendicular to plane *MN*. *JP* is also perpendicular to plane *MN* from the same point *J*. This is an impossibility; hence there can be only one common perpendicular between *AB* and *CD*.

§ 85. COROLLARY. **The shortest line that can be drawn between two skew lines is the common perpendicular.**

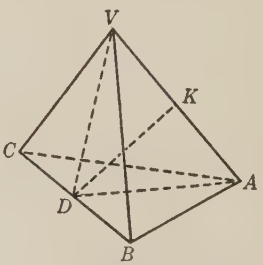


Let *EF* be the common perpendicular. Pass plane *MN* through *CD* and parallel to *AB*. Let *GH* be any other line between *AB* and *CD*. Draw *GK* perpendicular to *MN* and connect *K* with *H*. (The proof is left to the student.)

§ 86. The length of the perpendicular between two skew lines is called ***the distance between the skew lines.***

EXERCISE

1. *V–ABC* is a pyramid made with four equilateral triangles. *D* and *K* are the middle points of *CB* and *VA*, respectively. Prove that *DK* is perpendicular to *CB* and *VA*. (Prove $\triangle VDA$ isosceles.) If *AB* is 6 inches, how long is *DK*?



(Ex. 1)

Miscellaneous Exercises

Prove the following exercises:

1. If a line and a plane are parallel, a plane perpendicular to the line is perpendicular to the plane.

2. If two planes are perpendicular to each other, a line perpendicular to one of them and not contained in the other is parallel to the second plane.

3. If a line is parallel to a plane, a perpendicular to the plane from any point of the line is perpendicular to the line.

4. Tell how to construct a plane perpendicular to two intersecting planes through a point external to both.

5. Line AB is perpendicular to plane RS at B. PQ lies in RS but does not contain B. Show how to construct a plane which will contain AB and be perpendicular to PQ.

6. At any point F in altitude AD of $\triangle ABC$, FE is drawn perpendicular to the plane ABC. Through E, GH is drawn parallel to BC. Prove that AE is perpendicular to GH.

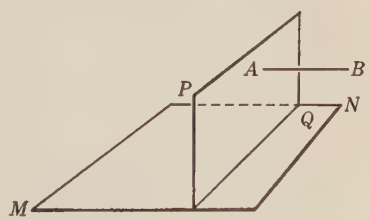
7. If three nonparallel planes are each perpendicular to a fourth plane, their intersections are parallel (or coincident).

8. If two intersecting lines are perpendicular, respectively, to two intersecting planes, the intersection of these planes is perpendicular to the plane determined by the first two lines.

9. If from a point within a dihedral angle perpendiculars are drawn to its faces, the angle formed by these lines is supplementary to the plane angle of the dihedral angle.

10. AA', BB', and CC' are parallel lines not all in the same plane. The plane PQ is perpendicular to CC'. Prove that PQ is perpendicular to the planes determined by the parallel lines.

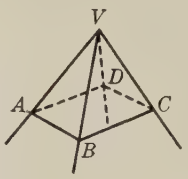
11. The plane PQ is perpendicular to the plane MN. $AB \perp PQ$ at A. Prove that any plane through AB intersects MN in a line perpendicular to PQ.



(Ex. 11)

Polyhedral Angles

§ **87.** The figure generated by a line which always intersects a fixed polygon and terminates in a fixed point not in the plane of the polygon is called a ***polyhedral angle.*** *V–ABCD* is a polyhedral angle. If there is no ambiguity, it may be read as polyhedral angle *V*; otherwise it must be read as above. The fixed point *V* is called the ***vertex.*** The moving line in any position is called an ***element.*** An element in the position *VA*, *VB*, etc., is called an ***edge.***



The parts of planes, such as *AVB*, *BVC*, etc., between two successive edges are called ***faces.*** (The faces are unlimited in extent. *ABCD* serves to generate the polyhedral angle but is not a part of it.) The angles, such as $\angle AVB$, $\angle BVC$, etc., between any two successive edges are called ***face angles.***

The dihedral angles, such as dihedral angles *VA*, *VB*, etc., formed by two successive faces are called ***dihedral angles of the polyhedral angle.***

§ **88.** A polyhedral angle is *convex* if the fixed polygon is convex. Only convex polyhedral angles are discussed in this book.

§ **89.** A polyhedral angle is called a ***trihedral angle***, ***tetrahedral angle***, etc., according as it has three, four, etc., faces.

§ **90.** Two polyhedral angles are called ***vertical polyhedral angles*** if the edges of one are the prolongations of the edges of the other.

§ **91.** A trihedral angle is ***rectangular***, ***birectangular***, or ***trirectangular*** according as it has one, two, or three right dihedral angles.

§ **92.** If a trihedral angle has two equal face angles, it is said to be ***isosceles.***

EXERCISES

1. Make a trihedral angle, using three equilateral triangles, each 3 inches on a side.

2. Make a tetrahedral angle, using four equilateral triangles.

3. Make a polyhedral angle, using five equilateral triangles.

4. Can you make a polyhedral angle, using six equilateral triangles? Give a reason for your answer.

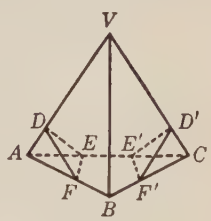
5. Cut out four congruent isosceles right triangles. Can you make a trihedral angle by using the right angles in three of these triangles? Can you make a tetrahedral angle by using the right angles of all four triangles?

6. Represent on paper two vertical trihedral angles.

7. Is it true that in a trirectangular trihedral angle the dihedral angles are all right angles? Why?

8. Draw three successive adjacent angles whose sizes are 100°, 65°, and 85°. Cut the figure from the paper and by folding on the sides of the angles make a trihedral angle.

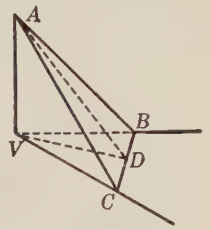
9. In trihedral angle V–ABC, $VA = VB = VC$ and $\angle AVB = \angle BVC$. $AD = CD'$. DE and DF are perpendicular to AV in faces AVC and AVB, respectively. $D'E'$ and $D'F'$ are perpendicular to VC in faces AVC and BVC, respectively. Prove that $\triangle AEF \cong \triangle CE'F'$. Prove that the dihedral angles VA and VC are equal.



(Ex. 9)

10. V–ABC is an isosceles trihedral angle in which $\angle AVB = \angle BVC$ and $VA = VB = VC$. Prove that $\triangle ABC$ is isosceles.

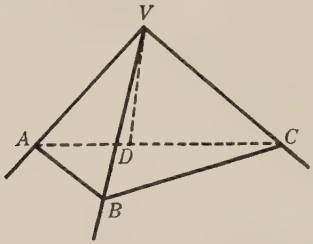
11. V–ABC is a trirectangular trihedral angle. $VA = VB = VC$. D is the middle point of CB. Prove that plane VAD is perpendicular to plane ABC.



(Ex. 11)

THEOREM 20

§ 93. The sum of any two face angles of a trihedral angle is greater than the third face angle.



Given V–ABC, a trihedral angle in which $\angle AVC$ is the greatest face angle.

To prove that $\angle AVB + \angle BVC > \angle AVC$.

Plan. $AB = AD$, $AB + BC > AC$; hence $BC > DC$. Then in $\triangle BVC$ and $\triangle DVC$, $\angle BVC > \angle DVC$ (Syllabus 67). But $\angle AVB = \angle AVD$; hence $\angle AVB + \angle BVC > \angle AVC$.

Proof. Draw AC. In the face AVC make $\angle AVD = \angle AVB$. Make $VB = VD$. Draw AB and BC.

The rest of the proof is left for you.

EXERCISES

1. To what plane geometry theorem does Theorem 20 correspond?

2. Prove: Any face angle of a polyhedral angle is less than the sum of the other face angles.

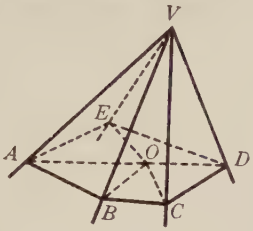
3. Prove: Any face angle of a trihedral angle is greater than the difference between the other two face angles.

4. What happens to a trihedral angle if a face angle is increased so that the sum of two face angles equals the third face angle?

5. The sum of the face angles of a trihedral angle is 280°. What is the greatest value that any one of the face angles may have?

THEOREM 21

§ 94. The sum of the face angles of any convex polyhedral angle is less than four right angles.



Given V–$ABCDE$ any convex polyhedral angle with n face angles. The sum of the angles at V is denoted by S.

To prove that $S < 4$ right angles.

Proof. Pass a plane which intersects the edges at A, B, C, D, and E. From any point O within the polygon $ABCDE$ draw lines to A, B, C, D, and E.

Let the sum of the angles at O be denoted by S'. Let the sum of the base angles of the triangles whose common vertex is V be denoted by K and the sum of the base angles of the triangles whose common vertex is O be denoted by K'. Then $S + K = 2n$ right angles, and $S' + K' = 2n$ right angles (Syllabus 53); hence $S + K = S' + K'$ (Axiom 1[1]). But $\angle VBA + \angle VBC > \angle ABC$ and a similar inequality exists at A, C, D, and E. (Why?) Hence $K > K'$. (Why?) Therefore $S < S'$ (Axiom 10). But $S' = 4$ right angles. (Why?) Therefore $S < 4$ right angles.

Preparation for Theorem 22. The following exercises will bring to mind certain plane geometry facts which will be helpful to you in Theorem 22. Prove each exercise.

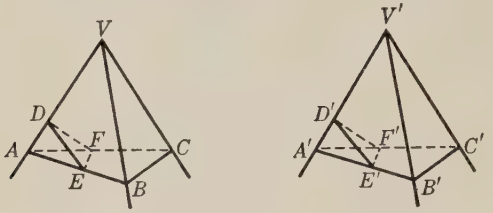
(1) A line perpendicular to one side of an acute angle and in the plane of the angle will meet the other side.

(2) A line perpendicular to one side of an obtuse angle and in the plane of the angle will not meet the other side, but will meet that side extended through the vertex of the angle.

[1] A list of axioms is given on pages 226–227.

THEOREM 22

§ 95. If two trihedral angles have the three face angles of one equal respectively to the three face angles of the other, the dihedral angles opposite the equal face angles are equal.



Given the trihedral angles V and V', with $\angle AVB = \angle A'V'B'$, $\angle BVC = \angle B'V'C'$, and $\angle CVA = \angle C'V'A'$.

To prove that the dihedral angles AV, BV, and CV are equal respectively to dihedral angles $A'V'$, $B'V'$, and $C'V'$.

Proof.

STATEMENTS	REASONS
1. Lay off $VA = VB = VC = V'A' = V'B' = V'C'$. Draw AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.	1. Syllabus 2.
2. $\angle AVB = \angle A'V'B'$.	2. Given.
3. $\triangle AVB \cong \triangle A'V'B'$.	3. Why?
4. $AB = A'B'$. Similarly, $BC = B'C'$ and $AC = A'C'$.	4. Why?
5. $\triangle ABC \cong \triangle A'B'C'$.	5. Why?
6. Lay off equal lines AD and $A'D'$ on AV and $A'V'$, respectively.	6. Syllabus 167.
7. At D draw $DE \perp AV$ in face AVB and $DF \perp AV$ in face AVC.	7. Syllabus 4.
8. $\angle VAB$ and $\angle VAC$ are acute.	8. Why?
9. DE and DF will meet AB and AC, respectively. Draw EF. Similarly, construct $\triangle D'E'F'$.	9. Why?
10. $\angle VAB = \angle V'A'B'$. $\angle VAC = \angle V'A'C$.	10. Why?
11. $\triangle ADE \cong A'D'E'$. $\triangle ADF \cong \triangle A'D'F'$.	11. Why?

The rest of the proof is left for you.

EXERCISES

Make models to show that the following statements are true. What theorem applies?

1. Polyhedral angles of three, four, and five faces, but not of six faces, may be formed by using angles of equilateral triangles as the face angles.

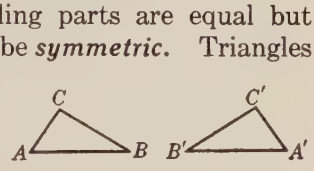
2. Polyhedral angles of three, but not four, faces may be formed by using right angles as face angles.

3. Polyhedral angles of three, but not four, faces may be formed by using angles of a regular pentagon as face angles.

4. A trihedral angle cannot be formed by using angles of a regular hexagon as face angles.

Symmetric and Congruent Figures

§ 96. Figures whose corresponding parts are equal but placed in opposite order are said to be ***symmetric.*** Triangles ABC and $A'B'C'$ shown here are symmetric; the angles and sides of one are equal, respectively, to the angles and sides of the other, but they are arranged in the reverse order. These two triangles cannot be made to coincide by any number of translations or rotations in the plane. (Cut two such triangles from cardboard and try to make them coincide without raising them from the plane of the table.) If, however, $\triangle ABC$ is rotated about AB as an axis until C falls again in the plane on the opposite side of AB, the two triangles can be made to coincide. Since the order of parts of a plane figure can always be reversed by a rotation out of the plane as illustrated above, *symmetric plane figures are also congruent.* Symmetric figures drawn in surfaces other than planes and symmetric solid figures are not in general congruent. (Cut two symmetric triangles from the surface of a rubber ball and try to make them coincide.)



§ 97. Two polyhedral angles are said to be ***congruent*** if they can be made to coincide.

§ 98. Two polyhedral angles are ***symmetric*** if the face angles and the dihedral angles of one are equal respectively to the face angles and the dihedral angles of the other, but arranged in the opposite order.

Are vertical polyhedral angles symmetric?

With the above definitions of *congruent* and *symmetric* polyhedral angles in mind, the following corollaries to Theorem 22 may readily be deduced:

§ 99. COROLLARY. **Two polyhedral angles are congruent if the face angles and the dihedral angles of one are equal respectively to the face angles and the dihedral angles of the other and arranged in the same order.**

§ 100. COROLLARY. **Two trihedral angles are congruent or symmetric if the three face angles of one are equal respectively to the three face angles of the other.**[1]

§ 101. It is interesting to note the analogy between triangles and trihedral angles. Consider, for example, the following two comparisons.

1. In a triangle the sum of any two sides is greater than the third side. In a trihedral angle the sum of any two face angles is greater than the third face angle.

2. In a triangle, if two sides are equal, the angles opposite those sides are equal. In a trihedral angle, if two face angles are equal, the opposite dihedral angles are equal.

The sides of a triangle are analogous to the face angles of a trihedral angle, and the angles of a triangle are analogous to the dihedral angles of a trihedral angle.

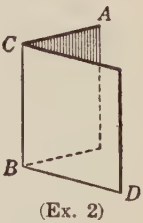
The same analogy exists between polygons and polyhedral angles. Write as many more of these corresponding statements as you can.

[1] Note that the proof of Theorem 22 is the same whether the corresponding parts are in the same order or in the opposite order.

Chapter Test

Complete statements 1–9.

1. The figure formed by two ? which ? each other is called a ? angle. The parts of such an angle are the ? and the ?.



(Ex. 2)

2. The angle in the figure may be read ? when there is no ambiguity. Otherwise it must be read ?.

3. The angle formed by two lines, one in each ? of a ? angle and ? to the ? at the ? ?, is called a ? angle of the ? angle.

4. All the ? angles of the same ? angle are ?.

5. The solid geometry theorem used in proving the statement of the preceding exercise is · · ·. (State in full.)

6. If two ? angles have a common ? and a ? ? ? them, they are called ? ? angles.

7. Two planes are said to be perpendicular to each other if they form a ? ? whose ? ? is a ? ?.

8. The measure of a dihedral angle is the measure of ? ? ?.

9. If two trihedral angles have three face angles · · ·.

10. How many dihedral angles has a cube? How many trihedral angles?

11. Is it possible to have a trihedral angle whose face angles are 90°, 30°, and 40°? Explain.

12. Is it possible to have a polyhedral angle with six face angles each 60°? Explain.

13. AB lies in plane MN. Plane RS is perpendicular to AB. Are planes MN and RS perpendicular? Explain.

14. AB is perpendicular to plane MN at B. Plane RS is perpendicular to MN and contains the point B. Does RS contain A? Explain.

15. Planes PQ and RS, intersecting in line AB, are both perpendicular to plane MN. At B, which is in MN, a line BC is drawn perpendicular to MN. What is the relation of BC to AB? Explain.

Miscellaneous Exercises

1. If two planes have a point in common, must they have more than one point in common?

2. Can there be more than one perpendicular from a line to a plane?

3. Must all the perpendiculars of Ex. 2 lie in a plane?

4. Can there be more than one perpendicular from a point to a plane?

5. If a line is perpendicular to a line of a plane, is it perpendicular to the plane?

6. If two parallel planes are cut by two other parallel planes, are all four intersections parallel?

7. If a line is perpendicular to two intersecting lines at their point of intersection, is it perpendicular to the plane of the lines?

8. Can a line be perpendicular to each of two planes?

9. If two planes are each perpendicular to a third plane, are they parallel?

10. If two lines are each perpendicular to the same plane, are they parallel?

Prove the following exercises:

11. If two face angles of a trihedral angle are equal, the opposite dihedral angles are equal.

12. In a trihedral angle whose three face angles are equal, the three dihedral angles are equal.

13. If the opposite face angles of a tetrahedral angle are equal, the opposite dihedral angles are equal.

14. If in a tetrahedral angle two adjacent face angles are equal to each other and the other two face angles are also equal to each other, the dihedral angle formed by the planes of two unequal face angles is equal to the dihedral angle formed by the planes of the other two unequal face angles.

15. The sum of the angles of a quadrilateral whose vertices are not all in the same plane is less than four right angles.

CHAPTER V

LOCUS AND PROJECTIONS

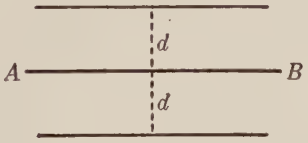
Loci in Two and Three Dimensions

Locus is a Latin word meaning *place*. In plane geometry a locus is usually a line or group of lines which contains all the points in a plane that satisfy a given condition, and which contains no points that do not satisfy the condition. In special cases the locus may be one point only — for example, the locus of points that satisfy the equation $x^2 + y^2 = 0$.

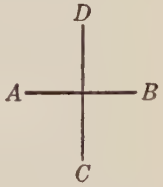
In three dimensional geometry a locus may be a surface as well as a line or lines. In a plane the locus of a point at a given distance from a given point is a circle, but in space the locus is a sphere (the sphere being considered as a surface). The locus of points common to two planes, as we know, is a straight line.

(1) Can you state the loci suggested by the figures below?

CIRCLE

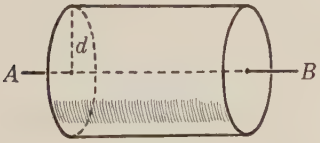


PARALLEL LINES

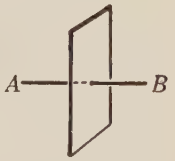


PERPENDICULAR BISECTOR

SPHERE



CYLINDER



PERPENDICULAR BISECTING PLANE

Locus

(1) What is the locus in a plane of a point equally distant from the extremities of a line of that plane? Note that this means: What line contains all the points which are in the plane and equally distant from the extremities of the line and no other points?

(2) What must be proved to show that the bisector of an angle contains all the points which are in the plane of the angle and equally distant from the sides of the angle? (Any point equally distant from the sides of the angle lies on the bisector of the angle.)

(3) What must be proved to show that the bisector of an angle contains no other points than those which are equally distant from the sides of the angle? (Any point on the bisector of an angle is equally distant from the sides of the angle.)

(4) Prove that the locus in a plane of the vertex of a right triangle with a given fixed hypotenuse as base is a circle upon the hypotenuse as diameter.

(5) Recall two other locus theorems from plane geometry and tell what it is necessary to prove in order to establish them. (If necessary, refer to the Syllabus, page 237.)

§ **102.** For a complete proof of a locus theorem it must be shown not only that the locus contains all the points which satisfy the given condition, but that it contains no other points. These two requirements may be stated as follows:

1. *Any point on the locus satisfies the condition.*

2. *Any point which satisfies the condition lies on the locus.*

It is sometimes convenient to prove the following statement instead of 2:

2 *a*. *Any point not on the locus does not satisfy the condition.*

Why is it justifiable to prove statement 2 *a* instead of 2? What relation does 2 *a* bear to 2?

EXERCISES [1]

1. What is the locus of a point which is 5 inches from a given point in a plane? in space?

2. What is the locus of a point which is equally distant from the extremities of a line in a plane? in space?

3. What is the locus of a point which is equally distant from all points on a circle in a plane? in space?

4. What is the locus of a point which is equally distant from the vertices of a triangle in a plane? in space?

5. What is the locus of a point which is 8 inches from a given line in a plane? in space?

6. What is the locus of a point which is equally distant from the faces of a dihedral angle?

7. What is the locus of a point which is —

(*a*) equidistant from two given parallel planes?

(*b*) equidistant from two given parallel planes and a distance *d* from a given point?

(*c*) equidistant from two given parallel planes and a distance *d* from a given plane?

(*d*) equidistant from two given points and lying on a given line?

(*e*) equidistant from two given points and a distance *d* from a given point?

(*f*) equidistant from two given points and equidistant from two given parallel planes?

(*g*) equidistant from two given points and equidistant from two given intersecting planes?

(*h*) equidistant from two intersecting planes and a distance *d* from a given point?

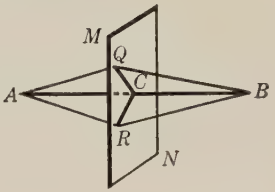
(*i*) equidistant from each of four given points not in the same plane?

(*j*) equidistant from the north and east walls of a room and from the floor?

[1] To the Teacher. Proofs are not to be required at this time.

THEOREM 23

§ 103. The locus of a point equally distant from the extremities of a line is the plane perpendicular to the line at its middle point.



Given plane MN perpendicular to AB at the middle point C.

To prove that MN is the locus of a point equally distant from A and B.

Plan. (1) Prove that any point Q in MN is equally distant from A and B. This will show that MN contains no points other than those equally distant from A and B.

(2) Prove that any point R which is equally distant from A and B lies in plane MN (Syllabus 30, § 31). This will show that MN contains all the points which are equally distant from A and B.

Proof. The proof is left for you.

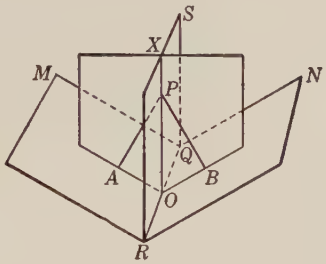
§ 104. COROLLARY. **The locus of a point equally distant from the vertices of a triangle is the line through the center of the circumscribed circle perpendicular to the plane of the triangle.**

The locus of a point equally distant from two of the vertices is a plane perpendicular to the line connecting these two points at its middle point. Likewise for two other vertices. The desired locus is the intersection of the two loci (§ 59). What is the intersection (§ 81)?

§ 105. A plane ***bisects a dihedral angle*** if it passes through its edge and makes equal angles with its faces.

THEOREM 24

§ 106. The locus of a point within a dihedral angle and equidistant from the faces of the dihedral angle is the plane that bisects the dihedral angle.



Given dihedral $\angle M\text{–}QR\text{–}N$ bisected by plane RS.

To prove (1) that any point P in plane RS is equidistant from planes MR and NR, and (2) that any point P that is equidistant from planes MR and NR lies in plane RS.

Proof of (1). Let P be any point on RS. Draw perpendiculars PA and PB from P to planes MR and NR respectively. Pass a plane AB through PA and PB, intersecting QR at O.

STATEMENTS	REASONS
1. Plane AB is $\perp$ to MR and to NR.	1. Why?
2. Hence QR is $\perp$ to plane AB.	2. Why?
3. Hence QR is $\perp$ to OA, OP, and OB.	3. Why?
4. Hence $\angle AOP$ and $\angle BOP$ are the plane angles of $\angle M\text{–}QR\text{–}S$ and $\angle N\text{–}QR\text{–}S$.	4. Why?
5. But $\angle AOP = \angle BOP$; i.e., OP bisects $\angle AOB$.	5. Why?
6. Hence $PA = PB$.	6. Why?
7. P is equidistant from MR and NR.	7. Why?

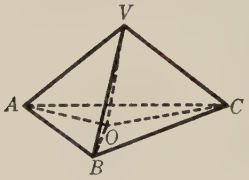
Proof of (2). Let P be any point equidistant from planes MR and NR. Pass a plane RS through P and RQ and prove that this plane bisects the dihedral angle.

$\angle AOP$ and $\angle BOP$ are the plane angles of $\angle M\text{–}QR\text{–}S$ and $\angle N\text{–}QR\text{–}S$ as above.

The rest of the proof is left for you.

THEOREM 25

§ 107. The bisecting planes of the dihedral angles of a trihedral angle intersect in a common line any point of which is equally distant from the three faces.



Given *V–ABC*, a trihedral angle in which the dihedral angles are bisected.

To prove that the bisecting planes meet in a common line any point of which is equally distant from the three faces.

Plan. The bisecting planes of the dihedral angles AV and BV will intersect, since they have the point V in common. Let the intersection be VO. Any point in VO is equally distant from the three faces. Why? Hence the bisecting plane of dihedral angle VC will contain VO (§ 106).

Proof. The proof is left for you.

EXERCISES

1. How would you find a point on a line and equally distant from the extremities of another line?

2. How would you find a point on a line and equally distant from the faces of a dihedral angle?

3. What is the locus of a point 5 inches from a given plane? What is the locus of a point at a given distance from a given plane?

4. What is the locus of a point 8 inches from a given point? What is the locus of a point at a given distance from a given point?

5. What is the locus of a point equally distant from two parallel planes?

6. What is the locus of a point equally distant from two intersecting planes?

7. What is the locus of a point equally distant from two points A and B and also equally distant from two other points P and Q? Discuss special cases.

8. What is the locus of a point equally distant from two points A and B and also at a given distance from a plane MN? Discuss special cases.

9. What is the locus of a point equally distant from two points A and B and also equally distant from two parallel planes MN and PQ? Discuss special cases.

10. What is the locus of a point equally distant from two points A and B and also equally distant from two intersecting planes? Discuss special cases.

11. What is the locus in a plane of a point at a given distance from a given point not in the plane? Prove that your statement is correct.

12. What is the locus in a plane of a point which is equally distant from two points not in the plane?

13. What is the locus of a point equally distant from two parallel lines? Prove that your statement is correct.

14. How would you find a point which is equally distant from four points not all in the same plane?

15. If a right triangle rotates about the hypotenuse as an axis, what is the locus of the vertex of the right angle? Prove that your statement is correct.

16. The right triangle ABC has its right angle at C. $AB = 5$ units; $AC = 3$ units. If the triangle rotates about AC as an axis, what is the locus of B? What are the length and area of the locus?

17. If in Exercise 16 the triangle rotates about AB, what is the locus of C? What are the length and area of the locus?

Projection

(1) In the sentence, "The photograph was projected on the screen," what is the meaning of the word *projected?* Is there a point on the screen corresponding to every point on the photograph? Note that the points on the screen are determined by rays of light coming from a common point and passing through each point of the photograph.

(2) When the sun is directly overhead, what is the nature of the shadow, on a horizontal plane, of a horizontal rectangle? (Assume that the rays of light from the sun are parallel.)

(3) If the source of light is a point directly above the middle point of a horizontal rectangle, will the shadow of this rectangle on a horizontal plane be larger or smaller than the rectangle?

§ **108.** ***Projection*** is a means of representing objects on surfaces (usually planes) by throwing forward on the surface all points of the object in the direction of lines drawn through them. The lines may be drawn from a point or they may be parallel to each other.

§ **109.** When the projecting lines are parallel, as in Figure 1, we have ***parallel projection.*** When the projecting lines pass through a point, as in Figure 2, we have ***central projection.***

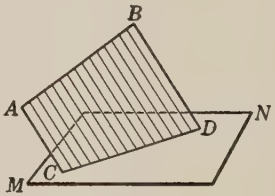


FIG. 1

In the case of parallel projection the lines may meet the plane at any angle, but often they are perpendicular to the plane. When the projecting lines are perpendicular to the plane, we have ***orthogonal projection.*** When we speak of projections in this book we refer only to orthogonal projections.

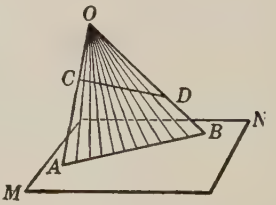


FIG. 2

§ **110.** The ***projection of a point on a plane*** is the foot of the perpendicular from the point to the plane. The perpendicular is called the ***projecting line*** of the point.

§ **111.** A plane which contains a given line and is perpendicular to a given plane is called the ***projecting plane*** of that line on the given plane.

(1) Prove that a point has only one projection on a given plane.

(2) May two points have the same projection on a plane?

(3) Represent a point and its projection on a horizontal plane. Represent the projection of the same point on a vertical plane. Prove that the plane determined by the two projecting lines is perpendicular to the intersection of the horizontal and the vertical plane.

§ **112.** The ***projection of a line on a plane*** (the line may be straight or curved) is the locus of the projections on the plane of all the points of the line.

§ **113.** The ***projection on a plane of a portion of a line*** (straight or curved) is that portion of the projection which lies between the projections of the extremities of the given portion of the line.

Preparation for Theorem 26. (1) If a line is perpendicular to a plane, what is its projection on the plane?

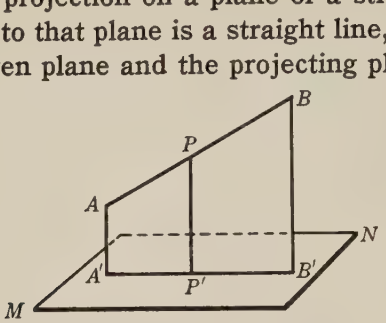
(2) Represent any line oblique to a given plane. Project on the plane ten points of the line and connect them.

(3) Bend a piece of wire into a curve which is not all in one plane. Fix it firmly above a piece of cardboard and project several of its points on the plane of the cardboard by means of thread. Draw a smooth curve through these points.

(4) What must be proved to establish Theorem 26? (Note that you do not take the locus and prove that it is a straight line; you take the straight line and prove that it is the locus.)

THEOREM 26

§ 114. The projection on a plane of a straight line not perpendicular to that plane is a straight line, the intersection of the given plane and the projecting plane.



Given any line AB not perpendicular to MN. AB', the projecting plane of AB, intersects MN in $A'B'$.

To prove that $A'B'$ is the projection of AB on MN.

Plan. (1) Prove that the projection of any point of AB lies in $A'B'$. (2) Prove that any point of $A'B'$ is the projection of a point of AB.

Proof. The proof is left for you.

EXERCISES

1. Prove: If a line is parallel to a plane, it is equal to its projection on that plane.

2. Prove: If two lines are parallel, their projections on the same plane are parallel or coincide.

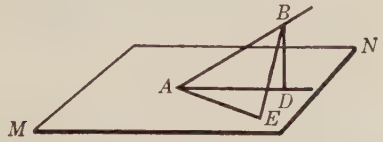
3. Prove: The projection of a parallelogram on a plane is a parallelogram (unless the plane of the parallelogram is perpendicular to the given plane).

4. Prove: The projections of a line segment upon two parallel planes are equal and parallel.

Preparation for Theorem 27. On a sheet of paper draw several lines through a point. Hold a pencil oblique to the plane of the paper, with its point at the intersection of these lines. Estimate the various angles formed by the pencil and the lines in the plane. Which is the smallest angle? Which is the largest?

THEOREM 27

§ 115. The acute angle which a straight line makes with its projection on a plane is the least angle which it makes with any line in that plane.



Given AD, the projection on MN of AB, which intersects MN at A. AE is any other line in MN through A.

To prove that $\angle BAD < \angle BAE$.

Plan. Draw the projecting line BD. Take $AE = AD$. (Use § 38, Syllabus 67.)

Proof. The proof is left for you.

§ 116. The angle which a straight line makes with its projection on a plane is called the angle which it makes with the plane, or its ***inclination*** to the plane.

§ 117. The ***projection of a figure on a plane*** is the figure formed by the projection on the plane of all the points of the given figure.

EXERCISES

1. Prove: If a line intersects two parallel planes, it makes equal angles with them.

2. Prove: The angle which a straight line makes with the prolongation of its projection on a plane is the largest angle which it makes with any line in that plane.

3. Prove: Equal oblique lines drawn to a plane from an external point make equal angles with the plane.

4. Represent a line AB 10 inches long meeting a plane MN at A at an angle of 30°. Draw the projecting line BD. Draw AD. What kind of triangle is formed? How long are AD and BD?

5. A line 10 inches long, if extended, would meet plane MN at an angle of 30°. How long is its projection on MN?

6. Two planes make an angle of 45° with each other. In one a line 8 inches long is drawn perpendicular to the intersection. How long is its projection on the other plane?

7. Two planes make an angle of 30° with each other. In one a line is drawn parallel to the intersection of the two planes. What is the relation of this line to its projection on the other plane?

8. Two planes intersect in a line AB. CD is drawn in one of them perpendicular to AB. Prove that the projection of CD on the other plane is also perpendicular to AB.

9. Prove: If one side of a right angle is parallel to a plane and the other side is not perpendicular to the plane, the projection of the angle on that plane is a right angle.

10. Two planes intersect in a line AB. In one of them a rectangle is drawn with one side AB. Prove that the projection of this rectangle on the other plane is also a rectangle.

11. Two planes meet in a line AB, making an angle of 60° with each other. In one of them a rectangle whose dimensions are 8×6 inches is drawn, the 6-inch side lying in AB. What is the area of the projection of this rectangle on the other plane?

12. If in the previous exercise the angle between the planes is 45° and the dimensions of the rectangle 3×5 inches, what is the area of the projection?

13. Two planes make an angle of 30° with each other. One plane contains a rectangle whose length is 10 inches and whose width is 7 inches. One side of the rectangle is parallel to the intersection of the two planes. What is the area of the projection of this rectangle on the other plane?

14. Answer the question of Exercise 13 when the angle between the planes and the length and the width of the given rectangle are as follows:

(*a*) 30°; a, b
(*b*) 45°; 9 in., 5 in.
(*c*) 45°; $3\frac{1}{2}$ yd., $1\frac{3}{4}$ yd.
(*d*) 45°; a, b
(*e*) 60°; 16 in., 9 in.
(*f*) 60°; 10.7 in., 8.95 in.
(*g*) 60°; a, b
(*h*) 60°; $a + b$, $a - b$

Areas of Projections by Trigonometry

The following exercises suggest a method of finding the areas of projections by using one of the trigonometric ratios.

(1) In $\triangle ABC$, $\angle C$ is a right angle. What is meant by $\sin B$, $\cos B$, $\tan B$, $\sin A$, $\cos A$, and $\tan A$?

(2) A line AB meets the plane MN and makes with it an angle of $a°$. Show that the projection of this line on MN equals $AB \cos a°$.

(3) A line AB would, if produced, make an angle of $a°$ with a plane MN. Show that the projection of this line on MN equals $AB \cos a°$.

(4) Two planes make an angle of $a°$ with each other. In one a rectangle of dimensions m inches and n inches is drawn so that one side is parallel to the intersection of the two planes. Show that the area of the projection of this rectangle on the second plane is $mn \cos a°$.

§ **118.** These exercises suggest the following proposition, which is proved in more advanced courses in mathematics:

The area of the projection on a plane of any plane figure is equal to the product of the area of the given figure and the cosine of the angle between the two planes.

EXERCISES

1. A line AB, 10 inches long, makes an angle of 20° with plane MN. How long is its projection on MN? (Use trigonometric tables.)

2. Answer the same question when AB and the angle are as follows: (*a*) 8.23 in., 25° 30′; (*b*) $10\frac{1}{2}$ in., 42° 20′; (*c*) 42.3 in., 21° 45′; (*d*) 6.43 in., 32° 26′.

3. The plane of a rectangle whose dimensions are 10×12 feet makes an angle of 35° with a second plane. What is the area of the projection of the rectangle on this plane?

4. The plane of a circle whose radius is 4 inches makes an angle of 50° with a second plane. What is the area of the projection of the circle on this plane?

Projective Geometry

The study of *projection* (see §§ 108–111) is an important specialized branch of mathematics called ***projective geometry.*** It has practical application in the study of perspective, the making of optical instruments, and map drawing.

A map is a representation (usually on a plane) of part or all of the earth's surface. If the earth were cylindrical, map making would be a simple matter. The map could be made on a cylindrical surface and then unrolled to form a plane surface. Since there is no way to open out a sphere into a plane without stretching the surface, making a map necessitates the projecting of points from a spherical surface to a plane. Many types of projection are used, depending upon the purposes for which the map is made. In all of them there is distortion of some sort. It is not possible, for example, to have equal lines, equal areas, and equal angles on the surface of the earth projected into equal lines, equal areas, and equal angles all on the same map.

Review Exercises

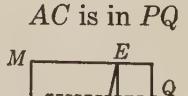
1. What is the locus of a point equally distant from two intersecting lines? Prove that your statement is correct.

2. What is the locus of a point equally distant from all the points of a given circle?

3. Prove: If three planes meet in the line AB, lines perpendicular to them from an external point lie in one plane.

4. The planes of two equilateral triangles ABC and ABC' make a dihedral angle of 90°. If AB is 10 inches, how long is CC'?

5. Plane MN is perpendicular to plane PQ. AC is in PQ and ED is in MN. EF and DF are perpendicular to AC. Prove that ED is perpendicular to PQ.



(Ex. 5)

6. Pass any plane through the base of an isosceles triangle and prove that the arms of the triangle make equal angles with the plane.

Chapter Test

1. What is the locus of a point which is —

(*a*) the bisector of all line segments joining two parallel lines?

(*b*) at a given distance from a given line?

(*c*) equidistant from two given points?

(*d*) equidistant from two intersecting planes?

(*e*) at a given distance from a given plane and at the same time equidistant from the extremities of a line segment oblique to the plane?

(*f*) equidistant from the faces of a dihedral angle and in a plane parallel to the edge of the angle?

(*g*) equidistant from two parallel lines?

(*h*) equidistant from the three vertices of a triangle?

2. A rectangle $ABCD$ rotates about AD as an axis. What is the nature of the surface generated by BC? If DC is 5 inches long, what is the length of the locus of C?

3. A rectangle 8 inches long and 6 inches wide is projected upon a parallel plane. Find the area of the projection.

4. What is the area of the projection of the rectangle in Exercise 3 upon a plane which makes an angle of 60° with the plane of the rectangle?

5. If the inclination to a plane of a line 16 inches long is 45°, what is the length of its projection on the plane?

6. The angle between two planes is 60°. A line in one plane, if extended, would be perpendicular to the intersection of the two planes. If the line is 5 inches long, how long is its projection on the other plane?

7. Two planes make an angle of 45°. In one of them a trapezoid $ABCD$ with altitude DE is drawn. AB, if extended, would be perpendicular to the intersection of the two planes. If AB, CD, and DE are a, b, and c inches, respectively, what are the lengths of their projections on the second plane? What is the area of the projected figure?

Organizing Your Knowledge

Make a list of:

1. Propositions concerning parallel lines in space.

2. Propositions concerning perpendicular lines in space.

3. Propositions concerning parallel planes.

4. Propositions concerning lines that are parallel to planes.

5. Propositions concerning perpendicular planes.

6. Propositions concerning lines that are perpendicular to planes.

7. Propositions involving loci in space.

8. Constructions which may be performed in space.

9. Propositions concerning dihedral angles.

10. Propositions concerning polyhedral angles.

11. Propositions concerning projections.

12. Ways of determining a plane.

13. Methods of proving that two planes are parallel.

14. Methods of proving that a line and a plane are parallel.

15. Methods of proving that two lines are parallel.

16. Methods of proving that two lines are perpendicular to each other.

17. Methods of proving that a plane and a line are perpendicular to each other.

18. Methods of proving that two planes are perpendicular to each other.

19. Methods of proving that a line lies in a plane.

20. All other propositions that have been discussed in Chapters I to V, but that do not come under any of the above headings.

CHAPTER VI

POLYHEDRONS

Definitions

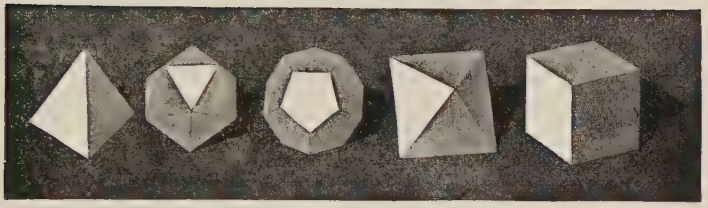


FIG. 1. REGULAR POLYHEDRONS

§ 119. A ***polyhedron*** is a solid completely bounded by planes. The portions of the planes bounding the polyhedron are called ***faces;*** the intersections of the faces are called ***edges;*** and the intersections of the edges are called ***vertices.*** A line joining any two vertices which are not in the same face is called a ***diagonal of the polyhedron.***

§ 120. A polyhedron with four faces is a ***tetrahedron;*** one with six faces is a ***hexahedron;*** one with eight faces is an ***octahedron;*** one with twelve faces is a ***dodecahedron;*** one with twenty faces is an ***icosahedron.***

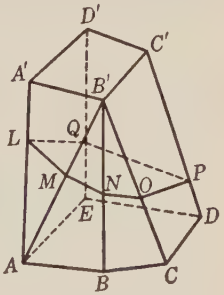


FIG. 2

§ 121. The figure formed by the intersection of a plane with a polyhedron is called a ***section*** of the polyhedron. In Figure 2 above *LMNOPQ* is a section of the polyhedron. If the section made by every plane cutting a polyhedron is a *convex polygon,* the solid is called a ***convex polyhedron.*** Only convex polyhedrons are considered in this book.

§ 122. The sum of the areas of all the faces of a polyhedron is called the ***total area of the polyhedron.***

EXERCISES

1. How many faces has the polyhedron shown in Fig. 2 on the preceding page? How many faces are triangles? How many are quadrilaterals? Name a polyhedral angle in this figure. Name a dihedral angle.

2. In the polyhedron in Exercise 1, how many sides has the section made by the plane LNP?

3. Represent a cube and a section of it made by a plane cutting only three of its faces.

4. What is the smallest number of faces a polyhedron may have?

5. Show that the faces of a polyhedron must be polygons.

6. Find in this book a figure which represents a polyhedron whose faces are all equal squares. How many edges has it? How many faces? What is it called?

7. Find in this book a figure which represents a polyhedron whose faces are all congruent equilateral triangles.

8. Find in this book a figure which represents a polyhedron whose faces are all equal regular pentagons. How many faces has it? How many edges has it?

9. Represent a polyhedron two of whose bounding planes are pentagons in parallel planes and whose other faces intersect in parallel lines. Do you know the name of the polyhedron you have represented?

10. Represent a polyhedron whose base is a regular hexagon in a horizontal plane and whose other faces meet at a point 8 inches directly above the center of the hexagon. Each side of the hexagon is 8 inches. Find the sum of the areas of all the faces of the polyhedron. What is the name of the polyhedron you have represented?

11. In the polyhedron of Exercise 10 represent a section parallel to the base and 4 inches from it. Find the perimeter and the area of this section. Find the sum of the areas of all the faces of the polyhedron that is formed when the top of the original polyhedron above the parallel section is cut off.

Regular Polyhedrons

§ 123. A *regular polyhedron* is a polyhedron all of whose faces are congruent regular polygons and all of whose polyhedral angles are equal. (See the drawings in Fig. 1 at the opening of this chapter.)

EXERCISES

1. Review Exercises 1–4, page 71.

2. Show that no more than three regular polyhedrons with triangular faces are possible.

3. Show that there can be no more than one regular polyhedron with square faces.

4. Show that there can be no more than one regular polyhedron with pentagonal faces.

5. Show that there can be no regular polyhedron with hexagonal faces.

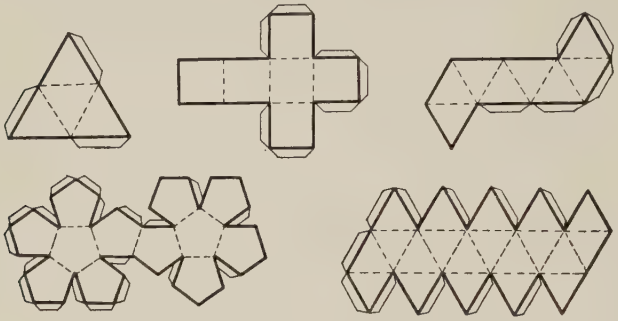
6. Do the previous exercises prove the existence of five regular polyhedrons?

7. The diagrams of Fig. 1 at the opening of this chapter are drawings of the five regular polyhedrons. Study these drawings and § 120. Then identify by name each of the polyhedrons pictured.

8. Find the total area of a regular tetrahedron each of whose edges is 4 inches.

§ 124. It does not follow from the previous exercises that it is possible to construct five regular polyhedrons. These exercises prove only that there cannot be more than five. As a matter of fact, it is possible to construct regular polyhedrons with polyhedral angles composed of three, four, or five equilateral triangles, three squares, or three regular pentagons. However, we shall not attempt a proof of this statement here.

§ **125.** If the following patterns are copied on cardboard, cut out, and folded along the dotted lines, they will form models of the five regular polyhedrons.



EXERCISES

1. How many faces has a regular polyhedron whose faces are equilateral triangles three of which meet at each vertex of the polyhedron? What is this polyhedron called?

2. How many faces has a regular polyhedron whose faces are equilateral triangles four of which meet at each vertex of the polyhedron? What is this polyhedron called?

3. How many faces has a regular polyhedron whose faces are equilateral triangles five of which meet at each vertex of the polyhedron? What is this polyhedron called?

4. How many faces has a regular polyhedron whose faces are squares?

5. How many faces has a regular polyhedron whose polyhedral angles are composed of three regular pentagons? What is this polyhedron called?

6. After you have made the models suggested above, verify the formula $F + V = E + 2$, where F is the number of faces of a polyhedron, V the number of vertices, and E the number of edges. Verify this formula for the other polyhedrons represented throughout this chapter.

Prisms

(1) Draw a plane pentagon $ABCDE$. Place the point of a pencil at A and, keeping the pencil always parallel to its first position, retrace the pentagon. Visualize the surface generated by the pencil.

(2) Represent a surface generated by a line which intersects a given polygon and is always parallel to a straight line not in the plane of the polygon.

(3) Draw a plane quadrilateral $ABCD$. Place the point of a pencil at A. Then, holding the pencil so that as nearly as possible it will always pass through a given point not in the plane, retrace the quadrilateral. Visualize the surface generated.

(4) Represent a surface generated by a line which intersects a polygon and passes through a point not in the plane of the polygon.

(5) Represent a surface generated by a line which intersects a circle and is always parallel to a line not in the plane of the circle.

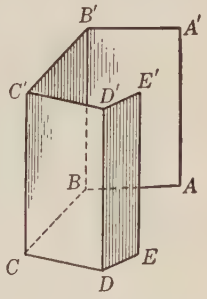


FIG. 1

§ 126. A ***prismatic surface*** is a surface generated by a straight line which always intersects a plane broken line and is always parallel to a given line not in the plane (Fig. 1). When the broken line is closed, the surface is called a ***closed prismatic surface.***

§ 127. A ***prism*** is a polyhedron bounded by a closed prismatic surface and two parallel planes intersecting the prismatic surface (Fig. 2). The polygons in the parallel planes are called the ***bases;*** the other faces are called the ***lateral faces;*** and the intersections of the lateral faces are called the ***lateral edges*** of the prism.

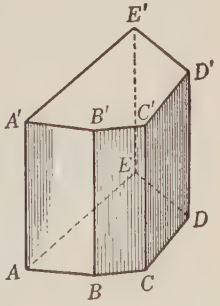


FIG. 2

§ 128. The *altitude of a prism* is the perpendicular distance between the parallel bases.

§ 129. A *triangular prism* is a prism with a triangle as base; a *quadrangular prism* is one with a quadrilateral as base; etc.

EXERCISES

1. Read the bases of the prism pictured in Fig. 2 on page 95. Read the lateral edges. What kind of figures are the lateral faces? Why?

2. Represent a triangular prism whose lateral edges are oblique to the base and also one whose lateral edges are perpendicular to the base.

3. Show that each side of one base of a prism is parallel to the corresponding side of the other base.

4. Prove: If one of the lateral edges of a prism is perpendicular to the base, they are all perpendicular to the base.

5. Represent a prism whose base is a square and whose lateral edges are oblique to the base and also one whose lateral edges are perpendicular to the base.

6. Two parallel planes cut all the lateral edges of a hexagonal prism. Prove that the sections are congruent hexagons.

7. In a certain triangular prism the lateral edges are perpendicular to the base, the sides of the triangle are 6, 6, and 8 inches respectively, and the length of the lateral edge is 10 inches. Find the area of each of the lateral faces and the total area of the prism.

8. In a certain prism the base is a rectangle 8×10 inches. The lateral edges, 6 inches long, are perpendicular to the base. Find the total area of the solid. Find the length of a diagonal of the solid. What name have you previously given to a solid like this one?

9. Prove: Corresponding diagonals of the two bases of a prism are parallel and equal.

Corollaries and Definitions

From the definitions which you have learned or are about to learn, the following corollaries are readily evident:

§ 130. COROLLARY. **The lateral edges of a prism are equal and parallel.**

§ 131. COROLLARY. **The lateral faces of a prism are parallelograms.**

§ 132. COROLLARY. **Sections of a prism formed by parallel planes cutting all the lateral edges are congruent.**

§ 133. COROLLARY. **The bases of a prism are congruent.**

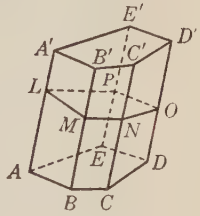


FIG. 1

§ 134. A *right section* of a prism is a section made by a plane which cuts all the lateral edges and is perpendicular to one of them. *LMNOP* is a right section.

§ 135. COROLLARY. **All the lateral edges of a prism are perpendicular to the plane of a right section.**

§ 136. COROLLARY. **The lateral edges of a prism are perpendicular to the sides of a right section.**

§ 137. COROLLARY. **All right sections of the same prism are congruent polygons.**

§ 138. A *right prism* is a prism in which the base is perpendicular to a lateral edge. Otherwise, it is an *oblique prism.*

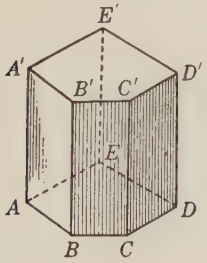


FIG. 2. RIGHT PRISM

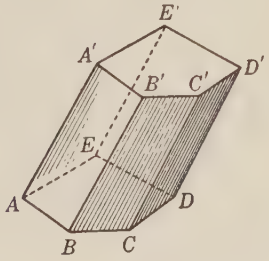


FIG. 3. OBLIQUE PRISM

§ 139. COROLLARY. **All the lateral edges of a right prism are perpendicular to both bases.**

§ 140. COROLLARY. **The lateral faces of a right prism are rectangles.**

§ 141. COROLLARY. **The altitude of a right prism is equal to a lateral edge.**

§ 142. A ***regular prism*** is a right prism whose base is a regular polygon.

§ 143. The ***lateral area*** of a prism is the sum of the areas of its lateral faces. The ***total area*** of a prism is the sum of the areas of its lateral faces and its bases.

EXERCISES

1. Represent a regular pentagonal prism.

2. Represent a right hexagonal prism and an oblique hexagonal prism with congruent bases and equal altitudes.

3. A right prism has an equilateral triangle for a base. Each side of the triangle is 6 inches and the altitude of the prism is 8 inches. What is the total area of the prism?

4. Answer the same question when each side of the triangle is a inches and the altitude of the prism is b inches.

5. Each side of the base of a regular hexagonal prism is 10 inches. The altitude of the prism is 8 inches. What is the total area?

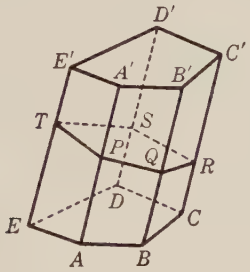
6. Answer the same question when the dimensions are a inches and b inches respectively.

7. An oblique prism has as a right section a pentagon whose sides are respectively 3, 4, 5, 6, and 7 inches long. The lateral edge of the prism is 7 inches long. What is the lateral area?

8. An oblique prism has as a right section a triangle whose sides are a, b, and c inches. The lateral edge of the prism is e inches. If $a + b + c = p$, find the lateral area of the prism in terms of e and p.

THEOREM 28

§ 144. The lateral area of a prism is equal to the product of a lateral edge and the perimeter of a right section.



Given a prism AD', with right section $PQRST$. The lateral area is denoted by L, the lateral edge by e, and the perimeter of $PQRST$ by p.

To prove that $L = ep$.

Proof.

STATEMENTS	REASONS
1. $AA' = BB' = CC' = \cdots = e$.	1. Why?
2. $PQRST$ is a right section.	2. Given.
3. $PQ \perp AA'$, $QR \perp BB'$, etc.	3. Why?
4. $AA'B'B$, etc., are parallelograms.	4. Why?
5. Area $\square AA'B'B = AA' \times PQ$, or $e \times PQ$.	5. Why?
6. Area $\square BB'C'C = e \times QR$, etc.	6. Why?
7. $L = e(PQ + QR + \cdots)$.	7. Why?
8. $PQ + QR + \cdots = p$.	8. Why?
9. $L = ep$.	9. Why?

EXERCISES

1. Find the lateral area of a prism whose lateral edge is 8 inches and the sides of a right section of which are 7, 8, 5, and 3 inches respectively.

2. Find the total area of a right prism whose altitude is 20 inches and whose base is an isosceles right triangle with hypotenuse equal to $15\sqrt{2}$ inches.

Parallelepipeds

§ 145. A *parallelepiped* is a prism whose bases are parallelograms.

EXERCISES

1. How many faces has a parallelepiped?

2. Prove: All the faces of a parallelepiped are parallelograms.

3. How many diagonals has a parallelepiped? Are all the diagonals equal?

4. Prove: Any lateral face of a prism determined by two successive lateral edges is parallel to the other lateral edges.

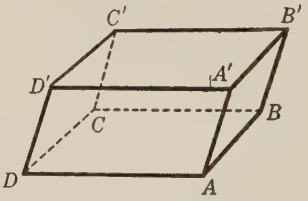
5. Prove: If a right section of a prism is a rectangle, the adjacent lateral faces are perpendicular to each other.

6. Prove: Any section of a prism made by a plane parallel to a lateral edge is a parallelogram.

7. Any two diagonals of a parallelepiped bisect each other.

Theorem 29

§ 146. The opposite faces of a parallelepiped are congruent and parallel.



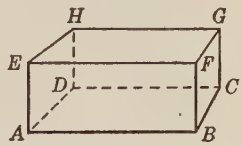
The proof is left for you.

§ 147. A *right parallelepiped* is one whose lateral edges are perpendicular to the bases.

§ 148. Corollary. A right parallelepiped is a right prism whose bases are parallelograms.

Rectangular Solids

§ 149. A ***rectangular solid*** (or *rectangular parallelepiped*) is a right parallelepiped whose bases are rectangles.



RECTANGULAR SOLID

§ 150. The lengths of the three edges meeting at any vertex of a rectangular solid are called its ***dimensions***, or its *length*, *width*, and *height*.

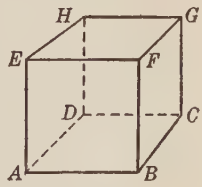
§ 151. In previous work, as was noted in § 3, a rectangular solid has been defined as a solid bounded by six rectangles. The definition above is more precise. From it you can at once deduce the fact that the faces are rectangles.

§ 152. COROLLARY. **All the faces of a rectangular solid are rectangles.**

§ 153. A ***cube*** is a rectangular solid whose dimensions are equal.

§ 154. COROLLARY. **All the edges of a cube are equal.**

§ 155. COROLLARY. **All the faces of a cube are squares.**



CUBE

(1) If two adjacent lateral faces of a parallelepiped are rectangles, is it a rectangular solid? Is it a right parallelepiped?

(2) If the lateral edges of a parallelepiped are perpendicular to the base, is it a rectangular solid?

(3) If the two bases and two opposite lateral faces of a parallelepiped are rectangles, is it a rectangular solid?

(4) Define prism, right prism, regular prism, parallelepiped, right parallelepiped, rectangular solid.

EXERCISES

1. If the dimensions of a rectangular solid are 6, 8, and 4 units, what is the total area?

2. Answer the same question for a rectangular solid whose dimensions are a, b, and c units.

3. Find the length of a diagonal of the rectangular solid in Exercise 1.

4. Find the length of a diagonal of the rectangular solid in Exercise 2.

5. Prove: The diagonals of a rectangular solid are equal.

6. If two of the dimensions of a rectangular solid are 8 and 12 inches respectively, and the diagonal is 16 inches, how long is the third dimension?

7. If two of the dimensions of a rectangular solid are a and b units respectively, and the diagonal is c units, how long is the third dimension?

8. If the edge of a cube is 8 inches, what is the length of the diagonal? What is the total area of the cube?

9. If the diagonal of a cube is 6 units, how long is an edge?

10. If the edge of a cube is e units, how long is the diagonal? What is the total area of the cube?

11. If the diagonal of a cube is a units, how long is an edge?

12. In a parallelepiped the three edges meeting at A are BA, EA, and DA; $\angle BAE = 60°$, $\angle EAD = 120°$, $\angle DAB = 90°$, $AB = 6$ inches, $AD = 8$ inches, and $AE = 4$ inches. Find the total area of the parallelepiped.

13. What is the sum of all the face angles of a parallelepiped?

14. How many square inches of copper lining are required to line the sides and bottom of a rectangular tank 20 inches long, 12 inches wide, and 8 inches high?

Review Exercises

1. Define polyhedron, regular polyhedron, tetrahedron, and hexahedron.

2. Name and describe the five regular polyhedrons.

3. What are the various parts of a polyhedron called? Illustrate.

4. What is a section of a polyhedron?

5. What are the particular properties of a prism which differentiate it from other polyhedrons?

6. Make a list of the properties of the general prism.

7. Differentiate between the lateral area and the total area of a prism.

8. What are the special properties of a right prism?

9. What are the special properties of a parallelepiped?

10. Is a parallelepiped whose bases are rectangles a right parallelepiped?

11. Is a parallelepiped whose lateral faces are all rectangles a right parallelepiped? Is it a rectangular solid?

12. The relationship between the faces, vertices, and edges of any polyhedron is given by the formula $F + V = E + 2$, where F is the number of faces, V the number of vertices, and E the number of edges. Using this formula, fill in the blanks in the table below.

NUMBER OF FACES, EDGES, AND VERTICES OF THE FIVE REGULAR POLYHEDRONS

NAME	F	E	V
Tetrahedron	4	6	
Cube	6	12	
Octahedron	8		6
Dodecahedron	12		20
Icosahedron	20		12

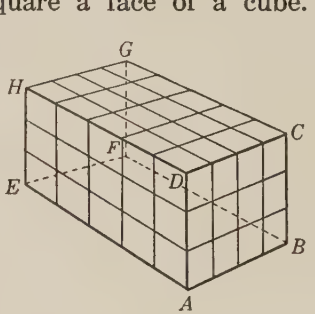
Volume

§ 156. A solid always encloses a portion of space. The ***volume*** of the solid is the *measure* of this space (not the space itself). It is the *number of units of space* it contains. The ***unit of volume*** is a cubic inch, cubic foot, cubic yard, etc. — in other words, a cube whose edge is of unit length.

When we say that the volume of a rectangular solid is 48 cubic inches, we mean that 48 cubes whose edges are each 1 inch will exactly fill the space enclosed by the solid.

§ 157. The formula $V = lwh$ for the volume of a rectangular solid is familiar to you. This means that, if you know the number of units in each of the three dimensions and if you find the product of these three *numbers* (not the units), you will get a *number* which shows how many cubic units are contained in the solid.

Figure *ABCD–EFGH* is a rectangular solid. *AB* is 4 units, *AD* is 3 units, and *AE* is 5 units. The face *ABCD* contains 12 square units, each square a face of a cube. Since *AE* is 5 units, there will be 5×12, or 60, cubic units in the solid. Multiplying $5 \times 4 \times 3$ gives the number of cubes contained in the solid — its volume.



If the dimensions of the solid were $6\frac{1}{2}$, $4\frac{1}{3}$, and $2\frac{1}{3}$ inches ($\frac{39}{6}$, $\frac{26}{6}$, $\frac{14}{6}$), a cube whose edges are each one sixth of an inch could be used as a unit. There would be $39 \times 26 \times 14$ such cubes. The number of cubic inches can be found by dividing this number by 216, the number of the smaller cubes contained in a cubic inch.

If the dimensions of the solid are *incommensurable lengths*, such as 3, $\sqrt{2}$, and $\sqrt{5}$ inches, an approximate volume may be found by extracting the square roots and using as a unit a cube whose edge is a common measure of these values. Any degree of approximation may be obtained.

§ 158. POSTULATE. **The volume of a rectangular solid is equal to the product of its three dimensions.**

If V denotes the volume of a rectangular solid, l its length, w its width, and h its altitude, then

$$V = lwh.$$

§ 159. COROLLARY. **The volume V of a cube is equal to the cube of an edge e.**

$$V = e^3.$$

The symbol = may be read "equal in volume." The context shows whether *equal* or *equal in volume* is meant.

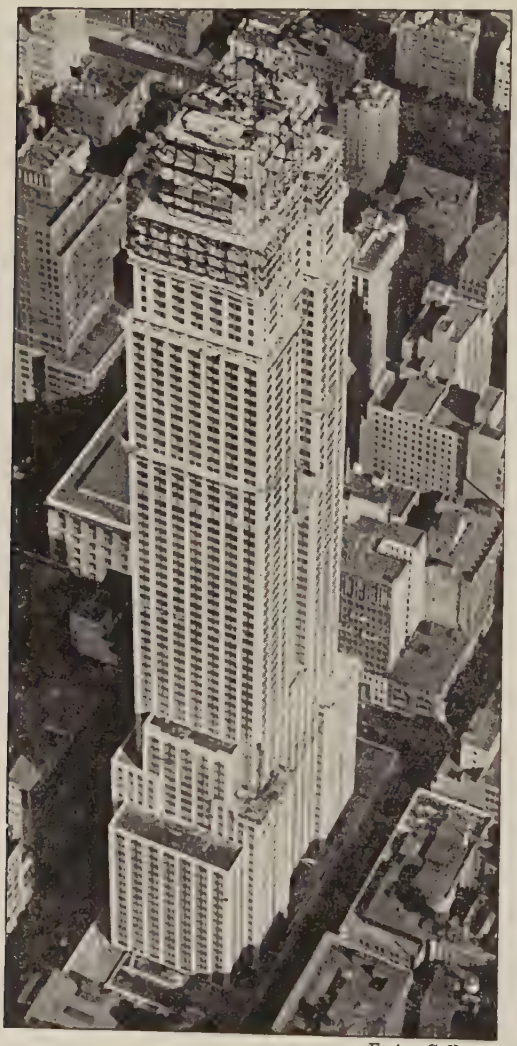
§ 160. COROLLARY. **The volume of a rectangular solid is equal to the product of its base [1] and its altitude.**

§ 161. ***Congruent solids*** are solids that can be made to coincide. Congruent solids are of course equal in volume, for, being superposable, they must contain the same number of cubic units. The converse statement is not necessarily true; solids which are equal in volume need not be congruent.

§ 162. COROLLARY. **Two solids composed of parts congruent each to each (not necessarily similarly placed) are equal in volume.**

§ 163. The measurement of areas and volumes was one of the earliest uses of mathematics, dating back at least to the time of the ancient Egyptians. The Egyptians knew how to find the area of a rectangle and a square, of course, because this involves nothing more than the conception of square measure. They had also solved correctly the problem of finding the area of a triangle, and their approximation to the area of a circle was good. The volume of a cube or rectangular solid was a simple matter to them, as it is to us, as a direct application of the meaning of volume. They went farther than this, however, and were able to find the volume of cylinders with considerable accuracy and had a formula for finding the volume of a square truncated pyramid.

[1] This means the number of square units in the area of the base.



Ewing Galloway

The picture above shows the Empire State Building, New York City, during its construction. One of the problems of the architect and the builder is the determination of the *volume* of a proposed building. In order to make a rough estimate of the cost of the building, the contractor multiplies the number of cubic feet by some constant denoting the approximate cost per cubic foot for that kind of building in his locality

EXERCISES

1. A classroom is 24 feet by 17 feet by 12 feet. Find the number of cubic feet of air space it contains.

2. Find the volumes of rectangular solids whose three dimensions are as follows:

(*a*) 10 in.; 8 in.; 5 in.
(*b*) $2\frac{1}{3}$ in.; $1\frac{3}{4}$ in.; $5\frac{1}{2}$ in.
(*c*) $10\frac{5}{8}$ in.; $6\frac{2}{3}$ in.; $4\frac{1}{2}$ in.
(*d*) 3.24 ft.; 2.62 ft.; 1.34 ft.
(*e*) a, b, c
(*f*) $a, a + 3, a - 2$
(*g*) $a, a - 2, a + 2$

3. Solve the formula $V = lwh$ for l.

4. Find the altitudes of rectangular solids having the following volumes and areas of bases:

(*a*) 50 cu. in.; 10 sq. in.
(*b*) 64 cu. in.; 15 sq. in.
(*c*) 8 cu. ft.; 8 sq. in.
(*d*) $25\frac{3}{4}$ cu. ft.; $5\frac{1}{2}$ sq. in.
(*e*) $30\frac{2}{3}$ cu. ft.; $6\frac{1}{4}$ sq. ft.
(*f*) 32.7 cu. ft.; 9.28 sq. ft.

5. What is the ratio of the volumes of two rectangular solids whose dimensions are 4, 3, and 2 units and 5, 6, and 2 units respectively?

6. What is the effect upon the volume of a rectangular solid when one dimension is doubled and the two other dimensions remain the same? when two dimensions are doubled? when all three are doubled?

7. What is the effect upon the volume of a rectangular solid if one dimension is doubled, another trebled, and the third multiplied by 4?

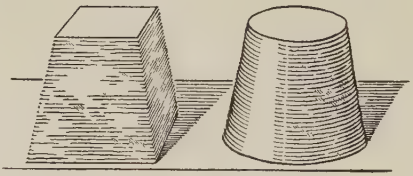
8. What is the effect upon the volume of a cube if each edge is increased in the ratio 1 : 4?

9. How long is each edge of a cube whose volume is twice the volume of a cube whose edges are each 3 inches?

10. Plot the equation $V = e^3$, thus showing the rapid rate of increase in the volume of a cube in comparison to the rate of increase of the edge.

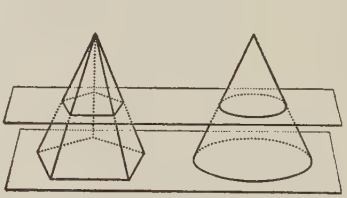
Cavalieri's Principle

§ **164.** The accompanying illustration shows a stack of 100 square cards and a stack of 100 round cards. All the cards are equal in thickness. The bottom two cards are equal in area. The two cards next to the bottom are also equal in area but are smaller than the bottom cards, etc., up to the top cards, which are also equal in area but are the smallest. Any card in the first stack, therefore, is matched by a card in the second stack the same height from the bottom and having the same area. The stacks, of course, are equal in volume.



Now we may think of any two solids as made up of very, very thin horizontal slices corresponding to thin cards, and if every such slice in one is matched by a slice in the other having the same area, the solids are equal in volume.

The two solids in the accompanying figure (called a *pyramid* and a *cone*) have bases of equal areas, and sections at equal heights from the bases are likewise equal in area. Comparing areas of sections is like comparing areas of corresponding slices. Hence we may assume the following principle, a more complete proof of which is given in higher mathematics:

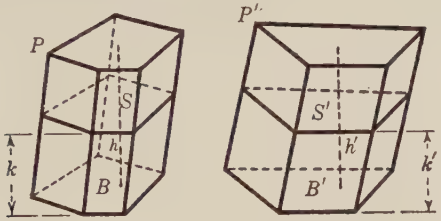


POSTULATE. **If two solids have equal altitudes, and if sections formed by planes parallel to the bases and at equal distances from the bases have equal areas, the solids are equal in volume.**

This proposition is known as ***Cavalieri's principle***, after the Italian mathematician Bonaventura Cavalieri (1598–1647), who first discovered it.

THEOREM 30

§ 165. Two prisms having equal bases and equal altitudes are equal in volume.



Given any two prisms P and P', with equal bases B and B' and equal altitudes h and h'.

To prove that $P = P'$.

Proof. Pass planes parallel to the bases of P and P', making sections S and S' at equal distances k and k' from the bases B and B', respectively.

STATEMENTS	REASONS
1. $S = B$ and $S' = B'$.	1. Why?
2. $B = B'$.	2. Given.
3. $S = S'$.	3. Why?
4. $h = h'$.	4. Given.
5. $k = k'$.	5. Construction.
6. Since S will always equal S' for any value of k, $P = P'$.	6. § 164.

§ 166. COROLLARY. **The plane passed through two diagonally opposite edges of a parallelepiped divides it into two equal triangular prisms.**

(1) Are two prisms with equal bases and equal edges equal? Give a reason for your answer.

(2) A side of the base of a regular triangular prism is 4 inches and its altitude is 6 inches. The dimensions of a rectangular solid are 2 inches, $2\sqrt{3}$ inches, and 6 inches. Prove that the volumes of the two solids are equal.

Review Exercises

1. Is there any difference between a regular tetrahedron and a regular triangular pyramid? Explain.

2. Can a plane be passed perpendicular to each of two intersecting lines at their point of intersection? Explain.

3. Planes MN and PQ are perpendicular. They intersect in AB. A plane RS intersects AB in O, MN in OC, and PQ in OD. Is angle COD a right angle? Explain.

4. Given a plane MN and a line AB not perpendicular to MN, show how to construct a line that shall be perpendicular to AB and parallel to MN. Prove your construction.

5. The diagonal of a rectangular solid is 6 inches. The base is a square and the altitude is one half the side of the base. Find the dimensions.

6. What is meant by the angle between a line and a plane? A line segment AB, 10 inches long, if extended will meet a plane at an angle of 25°. How long is the projection of this line on the plane? (Use trigonometric tables.)

7. Prove: The perpendicular from the intersection of the diagonals of a square upon a plane that does not cut the square is equal to one fourth the sum of the perpendiculars from the vertices of the square upon the same plane.

8. Planes MN and PQ are parallel. Plane RS is perpendicular to a line AB in plane MN. Prove that plane RS is perpendicular to plane PQ.

Preparation for Theorem 31. In connection with Theorem 31 it will be helpful at this point for you to review the following constructions from plane geometry:

(1) Transform a triangle into a right triangle.

(2) Transform a triangle into an isosceles triangle with a given base.

(3) Transform a pentagon into a triangle.

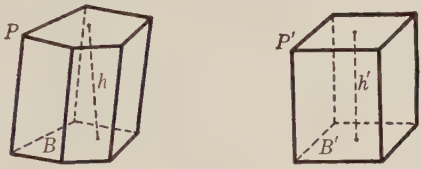
(4) Transform a hexagon into a triangle.

(5) Transform a triangle into a rectangle.

Is it possible to transform any convex polygon into a rectangle?

THEOREM 31

§ 167. The volume of any prism is equal to the product of its base and altitude.



Given any prism P, with base B and altitude h. Let V denote the volume.

To prove that $V = Bh$.

Plan. Construct a rectangular solid P' having a base B' equal in area to B and altitude h' equal to h. Denote its volume by V'. Show that $V' = V$ (§ 165). Since $V' = B'h'$, $h = h'$, and $B = B'$ then $V = Bh$.

Proof. The proof is left for you.

§ 168. COROLLARY. **The volume of a parallelepiped is equal to the product of the area of any face and the altitude upon that face.**

EXERCISES

1. Find the volume of a prism whose base is 24 square inches and whose altitude is 8 inches.

2. Find the volume of a triangular prism whose base is an equilateral triangle 10 inches on a side and whose altitude is 6 inches.

3. Find the volume of a triangular prism whose base is an equilateral triangle 16 inches on a side and whose altitude is 10 inches.

4. The altitude of a triangular prism is 6 inches, and its base is an isosceles right triangle whose arms are 8 inches each. Find the volume of the prism.

5. The base of a prism is an isosceles triangle whose arms are 5 inches and whose base is 4 inches. The altitude of the prism is 6 inches. Find the volume.

6. Find the volume of a triangular prism whose base is an equilateral triangle a inches on a side and whose altitude is b inches.

7. Find the volume of a triangular prism whose base is an isosceles right triangle with arms a inches each and whose altitude is b inches.

8. The base of a prism is an isosceles triangle whose arms are a inches each and whose base is b inches. The altitude of the prism is c inches. Find the volume.

9. Find the volumes of regular triangular, quadrangular, and hexagonal prisms whose bases are each 6 inches on a side and whose altitudes are 5 inches.

10. Find the volume of a regular hexagonal prism whose base is a inches on a side and whose altitude is b inches.

11. The base of a right prism is a triangle whose sides are 4, 13, and 15 inches respectively. The altitude of the prism is 6 inches. What is the volume?

12. If a cartload of earth is approximately a cubic yard, how many cartloads must be removed in digging a cellar $32 \times 20 \times 6$ feet?

13. How many gallons of water will a cistern $12 \times 10 \times 8$ feet hold? (A gallon is 231 cubic inches.)

14. If one ton of coal occupies 35 cubic feet, how many tons may be placed in a bin 10 feet long, 8 feet wide, and 6 feet deep?

15. Find the volume of a regular pentagonal prism whose base is 15.6 feet on a side and whose altitude is 8.34 feet. (Use trigonometric tables.)

16. Find the weight of water in a tank $10\frac{1}{2} \times 8\frac{3}{4} \times 6$ feet. (Water weighs $62\frac{1}{2}$ pounds per cubic foot.)

17. A swimming pool is 75 feet long and 25 feet wide. What is the weight of the water necessary to fill it to an average depth of $4\frac{1}{2}$ feet?

Pyramids

§ 169. A ***pyramidal surface*** is a surface generated by a straight line which always intersects a plane broken line and always passes through a point not in the plane of the broken line (Fig. 1). When the broken line is closed, the surface is called a ***closed pyramidal surface.*** Illustrate a pyramidal surface, using a yardstick.

FIG. 1

§ 170. The two parts of the pyramidal surface on each side of the point are called the ***nappes*** of the pyramidal surface (Fig. 1).

§ 171. A ***pyramid*** is a polyhedron bounded by one nappe of a closed pyramidal surface and a transverse plane — that is, a plane which intersects the nappe (Fig. 2).

§ 172. In the pyramid *V–ABCDE* (Fig. 2) *ABCDE* is the *base*, *V* is the *vertex*, *VA*, *VB*, *VC*, etc., are *lateral edges*, and *VAB*, *VBC*, etc., are *lateral faces.*

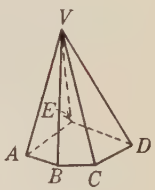


FIG. 2

§ 173. A pyramid is ***triangular***, ***quadrangular***, etc., according as its base is a *triangle*, a *quadrilateral*, etc.

§ 174. The ***altitude*** of a pyramid is the length of the perpendicular from the vertex to the plane of the base.

§ 175. A ***regular pyramid*** is a pyramid whose base is a regular polygon and whose altitude passes through the center of the base.

The ***axis*** of a regular pyramid is its altitude.

The ***slant height*** of a regular pyramid is the altitude of any one of its lateral faces.

Note that a regular tetrahedron (§§ 120, 123) is a regular pyramid all of whose faces are equilateral triangles. Any face of a regular tetrahedron may be considered as its base.

§ 176. COROLLARY. **The faces of a pyramid are triangles.**

§ 177. COROLLARY. **The lateral edges of a regular pyramid are equal.**

§ 178. COROLLARY. **The lateral faces of a regular pyramid are congruent isosceles triangles.**

§ 179. The ***lateral area of a pyramid*** is the sum of the areas of the lateral faces.

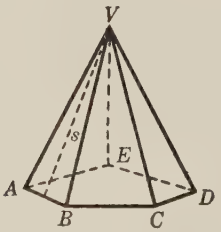
(1) A regular pyramid has for a base a square whose sides are each 8 inches. The lateral edges are each 6 inches. How long is the slant height? What is the lateral area? What is the total area?

(2) A regular pentagonal pyramid has a base whose sides are each a units. The slant height is b units. What is the lateral area?

(3) Repeat Exercise 2, substituting a regular octagonal pyramid.

THEOREM 32

§ 180. The lateral area of a regular pyramid is equal to half the product of the slant height and the perimeter of the base.



Given any regular pyramid, V–$ABCDE$, with slant height s, the perimeter of the base p, and lateral area L.

To prove that $L = \frac{1}{2}sp$.

Proof. The proof is left for you.

The Great Pyramids

Ewing Galloway

The great pyramids at Gizeh, in Egypt, built more than 6000 years ago, are among the outstanding man-made wonders of the world. These pyramids, built as tombs for the Egyptian kings, are in the form of regular quadrangular pyramids. The largest of these pyramids is sometimes referred to as the Great Pyramid. The sides of its square base are about 755 feet long; it has an altitude of about 480 feet; and its base covers an area of 13 acres.

EXERCISES

1. Find the lateral area of a regular triangular pyramid each side of whose base is 8 inches and whose slant height is 5 inches.

2. The altitude of a regular quadrangular pyramid is 4 inches. Each side of its base is 6 inches. Find its lateral area.

3. Find the lateral area of a regular hexagonal pyramid each side of whose base is 6 inches and whose lateral edge is 8 inches.

4. The altitude of a regular hexagonal pyramid is 8 inches. Each side of its base is 6 inches. What is its lateral area?

Truncated Solids — Frustums

§ **181.** A ***truncated prism*** is a part of a prism included between a base and a section made by a plane not parallel to the base and cutting all the lateral edges (Fig. 1). What is the shape of the lateral faces of a truncated prism?

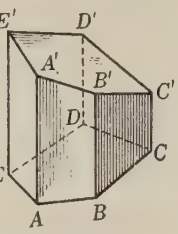


FIG. 1

§ **182.** A ***truncated pyramid*** is a part of a pyramid included between the base and a plane cutting all the lateral edges (Fig. 2).

§ **183.** A ***frustum of a pyramid*** is a part of a pyramid included between the base and a plane parallel to the base (Fig. 3). The base of the pyramid and the section made by the plane parallel to it are called the ***bases of the frustum.*** The other faces are called the ***lateral faces.***

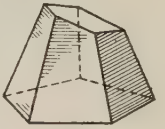


FIG. 2

§ **184.** The ***altitude of a frustum*** of a pyramid is the perpendicular distance between the bases.

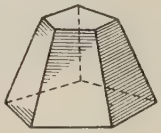


FIG. 3

§ **185.** COROLLARY. **The lateral faces of a frustum of a pyramid are trapezoids.**

§ **186.** COROLLARY. **The lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids.**

§ **187.** The ***slant height of a frustum of a regular pyramid*** is the altitude of any one of the lateral faces.

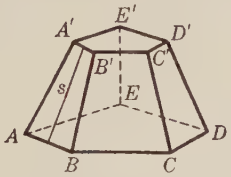
§ **188.** The ***lateral area of a frustum of a pyramid*** is the sum of the areas of the lateral faces.

(1) The lower base of the frustum of a regular pyramid is a square 4 inches on a side; the upper base is 3 inches on a side. The slant height is 2 inches. Find its lateral area.

(2) What is the lateral area of the frustum of a regular pyramid if the perimeter of the lower base is p, the perimeter of the upper base p', and the slant height s?

THEOREM 33

§ 189. The lateral area of a frustum of a regular pyramid is equal to half the sum of the perimeters of the bases multiplied by the slant height of the frustum.



Given a frustum of a regular pyramid with lateral area L, slant height s, and the perimeters of the bases p and p'.

To prove that $L = \frac{1}{2}\, s(p + p')$

Proof. The proof is left for you.

EXERCISES

1. The bases of a frustum of a regular pyramid are squares 8 inches and 5 inches on a side. The altitude of the frustum is 6 inches. What is the lateral area?

2. If each side of the base of a regular quadrangular pyramid is 8 feet and the slant height is 10 feet, what is the altitude?

(Ex. 2)

3. The lateral edge of a regular tetrahedron is 8 units. How long is the altitude?

4. If the altitude of a regular quadrangular pyramid is 6 feet and the slant height is 10 feet, what is the area of the base?

5. The altitude of a regular tetrahedron is 16 units. What is its total area?

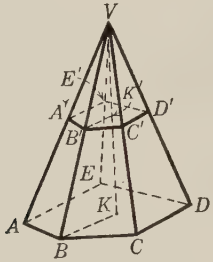
6. The altitude of a regular hexagonal pyramid is 8 inches and a side of the base is 6 inches. What is the lateral area of the frustum formed by a plane parallel to the base and 4 inches from it? What is the total area of the frustum?

THEOREM 34

§ 190. If a pyramid is cut by a plane parallel to its base:

(*a*) The lateral edges and the altitude are divided proportionally.

(*b*) The section formed is a polygon similar to the base.



Given pyramid V–$ABCDE$, cut by a plane parallel to its base, intersecting the lateral edges in A', B', C', D', and E' and the altitude VK in K'.

To prove (*a*) $\dfrac{VA'}{VA} = \dfrac{VB'}{VB} = \cdots = \dfrac{VK'}{VK}$.

(*b*) $A'B'C'D'E' \sim ABCDE$.

Proof. (*a*) Pass a plane through V parallel to $ABCDE$. (Use § 61.)

(*b*) The corresponding angles are equal (§ 60).

$\triangle VA'B' \sim \triangle VAB$; $\triangle VB'C' \sim \triangle VBC$. (Why?)

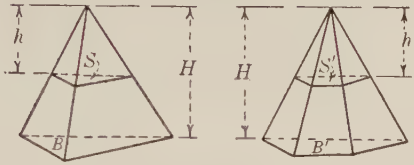
$\dfrac{A'B'}{AB} = \dfrac{VB'}{VB}$, $\dfrac{B'C'}{BC} = \dfrac{VB'}{VB}$; hence $\dfrac{A'B'}{AB} = \dfrac{B'C'}{BC}$.

§ 191. COROLLARY. **A section of a pyramid parallel to the base is to the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.**

$\dfrac{A'B'C'D'E'}{ABCDE} = \dfrac{\overline{A'B'}^2}{\overline{AB}^2}$; $\dfrac{A'B'}{AB} = \dfrac{VB'}{VB} = \dfrac{VK'}{VK}$. (Syllabus 126.)

$\dfrac{A'B'C'D'E'}{ABCDE} = \dfrac{\overline{VK'}^2}{\overline{VK}^2}$.

§ 192. COROLLARY. **If two pyramids have equal altitudes and equal bases, sections parallel to the bases at equal distances from the vertices are equal.**



$$\frac{S}{B} = \frac{h^2}{H^2}, \quad \frac{S'}{B'} = \frac{h^2}{H^2}, \quad \frac{S}{B} = \frac{S'}{B'}.$$

But $B = B'$; therefore $S = S'$.

EXERCISES

1. The altitude of a pyramid is 10 inches, the base is a square whose sides are each 6 inches.

(*a*) Find the area of a section parallel to the base and at a distance of 5 inches from the vertex.

(*b*) Find the area of a section parallel to the base and 3 inches from it.

2. The altitude of a pyramid is a inches, the base is a square whose sides are each b inches.

(*a*) Find the area of a section parallel to the base and c inches from the vertex.

(*b*) Find the area of a section parallel to the base and c inches from it.

3. The altitude of a pyramid is 12 feet. The base is a triangle whose sides are 9, 10, and 17 feet. Find the area of a section parallel to the base and 4 feet from the vertex.

4. The base of a regular quadrangular pyramid is 8 inches on a side. The altitude is 10 inches. How far from the vertex must a plane be passed to form a section whose area is 24 square inches?

5. The area of the base of a pyramid is B, the altitude is h. How far from the vertex must a plane be passed to form a section which is equal to half the base?

6. If a section parallel to the base of a pyramid is halfway between the vertex and the base, what is the ratio of the section to the base? What is the ratio if a section is one third of the distance from the vertex to the base?

7. If the distance of a section from the vertex of a pyramid is doubled, what is the effect upon the area of the section? What is the effect if the distance is trebled?

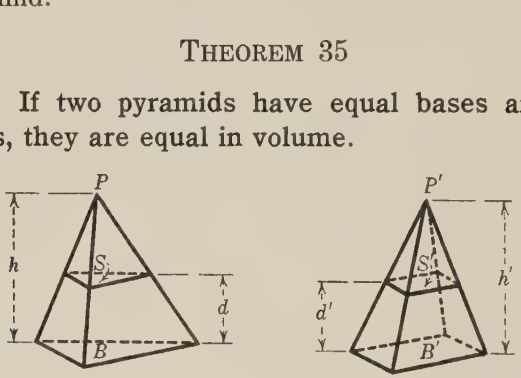
8. How far from the vertex of a pyramid must a plane be passed to form a section that is equal to one third the base?

9. The bases of a frustum of a regular pyramid are squares whose sides are each 4 inches and 3 inches respectively. The altitude of the frustum is 1 inch. Find the altitude of the pyramid.

10. The bases of a frustum of a regular pyramid are squares whose sides are a units and b units respectively. The altitude of the frustum is c units. Find the altitude of the pyramid.

Theorem 35

§ 193. If two pyramids have equal bases and equal altitudes, they are equal in volume.



Given pyramids P and P', with equal altitudes h and h' and equal bases B and B'.

To prove that $P = P'$.

Proof. Pass planes parallel to the bases at equal distances d and d' from each base, making sections S and S'. Then $S = S'$ (§ 192). Since this is true for any value of d less than h, $P = P'$ (§ 164).

Preparation for Theorem 36

(1) The altitude of a regular quadrangular pyramid is 10 inches and the sides of its base are each 8 inches. Another pyramid has the same altitude and a base which is an isosceles right triangle whose hypotenuse is 16 inches. Show that the two pyramids are equal in volume.

(2) If one of two pyramids which have equal altitudes has a square base 16 inches on a side and the other an equilateral triangle for a base, what must be the length of each side of the equilateral triangle in order that the volumes of the two pyramids shall be equal?

(3) The base of pyramid *V–ABCD* is a square. What part of pyramid *V–ABCD* is the pyramid *V–ABD*?

(4) The base of pyramid *V–ABCDEF* is a regular hexagon. (The vertices of the hexagon are lettered in succession.) What part of pyramid *V–ABCDEF* is pyramid *V–ABC*? pyramid *V–ACE*? (Draw the radii *OA*, *OC*, and *OE* of the hexagon.)

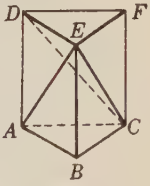


FIG. 1

(5) *ABC–DEF* is a regular triangular prism whose altitude is 10 inches and each side of whose base is 6 inches. Pass a plane through *EA* and *EC*. What kind of solid is *E–ABC*? What is the area of its base? What is its altitude?

(6) In the same figure pass a plane through *EC* and *DC*. Show that the volume of pyramid *C–DEF* is equal to that of pyramid *E–ABC*.

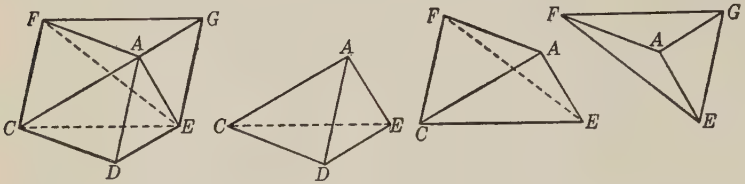
(7) In the same figure what two pyramids have their vertices at *E* and their bases in the same plane? Show that these two pyramids are equal in volume.

(8) In the same figure compute the volume of pyramid *E–ABC*.

(9) Show that any triangular prism may be divided into three equal pyramids.

Theorem 36

§ 194. The volume of a triangular pyramid equals one third the product of its base and altitude.



Given a triangular pyramid A–CDE, with volume V, base B, and altitude h.

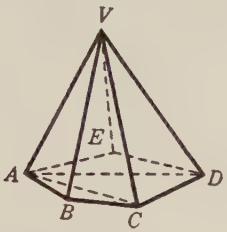
To prove that $V = \frac{1}{3}Bh$.

Proof.

Statements	Reasons
1. Construct the prism CDE–FAG with CDE as base and AD as a lateral edge.	1. Show how this is done.
2. CDE–$FAG = A$–$CDE + A$–$CEGF$.	2. Why?
3. Pass a plane through AE and FE, dividing A–$CEGF$ into A–CEF and A–EGF.	3. § 13.
4. $CEGF$ is a parallelogram.	4. Why?
5. $\triangle CEF = \triangle EGF$.	5. Why?
6. The altitudes of A–CEF and A–EGF are the same.	6. Why?
7. A–$CEF = A$–EGF.	7. § 193.
8. A–$EGF = E$–AGF.	8. Identity.
9. $\triangle CDE = \triangle AGF$.	9. Why?
10. The altitudes of E–AGF and A–CDE are equal.	10. Why?
11. E–$AGF = A$–CDE.	11. § 193.
12. The triangular pyramids, A–CDE, E–AGF, and A–CEF, are equal.	12. Why?
13. The sum of the volumes of A–CDE, E–AGF, and A–CEF is equal to the volume of the prism.	13. Why?
14. The volume of the prism $= Bh$.	14. § 167.
15. $V = \frac{1}{3}Bh$.	15. Why?

THEOREM 37

§ 195. The volume of any pyramid equals one third the product of its base and altitude.



The plan is to divide the pyramid into triangular pyramids as shown.

The proof is left for you.

EXERCISES

1. Find the volumes of pyramids whose bases and altitudes are as follows:

(*a*) 10 sq. in.; 3 in.
(*b*) 25 sq. in.; 4 in.
(*c*) $10\frac{1}{2}$ sq. in.; $5\frac{3}{4}$ in.
(*d*) 3 sq. ft.; 8 in.
(*e*) 9.24 sq. ft.; 2.62 ft.
(*f*) 62.43 sq. ft.; 24.82 ft.
(*g*) a^2; b
(*h*) a^2; $a + 3$
(*i*) $(a + 2)^2$; $a + 2$

2. Solve $V = \frac{1}{3}Bh$ for B and for h.

3. Find the altitudes of pyramids whose volumes and bases are as follows:

(*a*) 50 cu. in.; 25 sq. in.
(*b*) 3 cu. ft.; 2 sq. ft.
(*c*) $20\frac{3}{4}$ cu. ft.; $10\frac{1}{2}$ sq. ft.
(*d*) 32.3 cu. in.; 20.7 sq. in.
(*e*) 62.35 cu. in.; 43.56 sq. in.

4. A pyramid is formed by connecting the center of the upper base of a cube with the vertices of the lower base. If each side of the cube is 6 inches, what are the volume and the lateral area of the pyramid?

5. The edges of a regular octahedron are 5 inches long. Find its volume.

EXERCISES

1. Find the total areas of regular triangular pyramids of the following dimensions:

(*a*) Altitude, 8 inches; lateral edge, 10 inches
(*b*) Base edge, 6 inches; altitude, $\sqrt{22}$ inches
(*c*) Base edge, 8 inches; lateral edge, 5 inches
(*d*) Lateral edge, 10 inches; altitude of base, 5 inches

2. Find the volumes of the pyramids in Exercise 1.

3. Find the volumes of regular triangular pyramids of the following dimensions:

(*a*) Lateral edge, 15 inches; slant height, 12 inches
(*b*) Lateral edge, a inches; slant height, b inches

4. Find the lateral area and the volume of a regular quadrangular pyramid whose altitude is 5 inches and whose slant height is 13 inches.

5. Find the lateral area and the volume of a regular hexagonal pyramid whose lateral edge is 7 inches and whose base edge is 4 inches.

6. The area of the base of a pyramid is 81 square inches. The altitude of the pyramid is 6 inches. What is the area of a section of the pyramid parallel to the base and cutting the altitude 4 inches from the vertex?

7. Find the altitude of a regular quadrangular pyramid whose volume is 160 cubic inches and whose base edge is 10 inches.

8. The lateral edge of a regular pyramid whose base is 40 square inches is 8 inches long and is inclined at an angle of 60° to the plane of the base. Find the volume of the pyramid.

9. A regular hexagonal pyramid whose lateral edge is 15 inches is cut by a plane parallel to the base and 5 inches from the vertex. If each side of the base is 6 inches, how long is each side of the section? Find the area of the section in two ways.

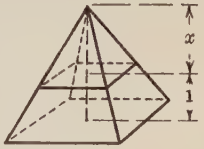
Volume of a Frustum of a Pyramid

§ 196. The following exercises will suggest to you a method for finding the volume of a frustum of a pyramid.

(1) The areas of the bases of a frustum of a pyramid are 36 and 25 square inches respectively. Its altitude is 1 inch. Find the altitude of the pyramid.

SUGGESTION. Let x be the altitude of the part of the pyramid cut off by the upper base. Then the altitude of the pyramid is $x + 1$. By § 191,

$$\frac{(x+1)^2}{x^2} = \frac{36}{25}; \quad \frac{x+1}{x} = \frac{6}{5}.$$



(2) Find the volume of the frustum of Exercise 1.

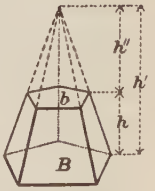
(3) The areas of the bases of a frustum of a pyramid are 27 and 12 square inches. Its altitude is 2 inches. Find the altitude of the pyramid. Find the volume of the frustum.

(4) The areas of the bases of a frustum of a pyramid are 40 and 24 square inches. Its altitude is 2 inches. Find its volume.

(5) Prove that the volume of a frustum of a pyramid is expressed correctly by the formula

$$V = \tfrac{1}{3}h(B + b + \sqrt{Bb}),$$

in which B and b are the bases and h is the altitude.



SUGGESTION. Express the volume as the difference between the volume of the whole pyramid and the volume of the part cut off by the upper base. Let V, V', V'', and h, h', h'', be the volumes and the altitudes of the frustum, pyramid, and part removed respectively. Let B and b be the lower base and the upper base of the frustum. Then $h'' = h' - h$. Now $V = V' - V'' = \frac{1}{3}[Bh' - b(h' - h)]$. $V = \frac{1}{3}[h'(B - b) + bh]$, but $\frac{B}{b} = \frac{h'^2}{(h'-h)^2}$ or $\frac{\sqrt{B}}{\sqrt{b}} = \frac{h'}{h'-h}$. Solve for h' and substitute in the equation for V above. Remember that $B - b = (\sqrt{B} + \sqrt{b})(\sqrt{B} - \sqrt{b})$.

Similar Polyhedrons

§ **197.** Polyhedrons that have the same number of faces similar each to each and similarly placed and that have their corresponding polyhedral angles equal are called ***similar polyhedrons.***

(1) Show that any two cubes are similar polyhedrons.

(2) Show that in two cubes whose edges are a and b respectively (a) the diagonals have the same ratio as the edges, (b) the total areas are to each other as the squares of the edges, and (c) the volumes are to each other as the cubes of the edges.

(3) Are all rectangular solids similar?

(4) Show that two rectangular solids whose corresponding edges are proportional are similar polyhedrons. Show also that in two similar rectangular solids (a) the diagonals have the same ratio as any two corresponding edges, (b) the total areas are to each other as the squares of corresponding edges, and (c) the volumes are to each other as the cubes of corresponding edges.

SUGGESTION. If the dimensions are a, b, c and a', b', c' respectively, and $a : a' = b : b' = c : c'$, then $a' = ka$, $b' = kb$, and $c' = kc$, where k is the factor of proportionality.

(5) Show that two regular hexagonal prisms whose lateral edges have the same ratio as the edges of the bases are similar polyhedrons. Show also that (a) the total areas are to each other as the squares of corresponding edges and (b) the volumes are to each other as the cubes of corresponding edges.

(6) Show that two pyramids whose bases are similar and whose lateral edges have the same ratio as the base edges are similar polyhedrons. Show also that the total areas are to each other as the squares of any two corresponding edges, and that the volumes are to each other as the cubes of any two corresponding edges.

The relationships which you have noted in your work on the preceding page with particular similar polyhedrons are also properties of similar polyhedrons in general. We state these relationships here as postulates, without proof.

1. In similar polyhedrons any two corresponding lines have the same ratio as any other two corresponding lines.

2. In similar polyhedrons the total areas or any corresponding areas are to each other as the squares of any two corresponding lines.

3. In similar polyhedrons the volumes are to each other as the cubes of any two corresponding lines.

EXERCISES

1. The total area of a polyhedron, one of whose edges is 10 inches, is 540 square inches. What is the total area of a similar polyhedron whose corresponding edge is 12 inches?

2. In what ratio must the edges of a polyhedron be increased in order to obtain a similar polyhedron with three times the total area?

3. If it costs a dollars for paint to cover a certain garage, how much will it cost for paint for a similar garage whose dimensions are three fourths as much?

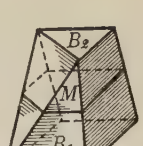
4. The volume of a polyhedron one of whose edges is 6 inches is 720 cubic inches. Find the length of the corresponding side of a similar polyhedron whose volume is 480 cubic inches.

5. In what ratio must the edges of a polyhedron be increased in order to obtain a similar polyhedron whose volume is eight times as great? twice as great?

6. The total areas of two similar polyhedrons are 36 square inches and 64 square inches, respectively. If the volume of the first is 12 cubic inches, what is the volume of the second?

7. A pyramid has an altitude of 12 inches. How far from the vertex must a plane be passed parallel to the base in order to divide the pyramid into two equal parts?

Prismatoids



§ 198. A polyhedron all of whose vertices lie in two parallel planes is called a ***prismatoid.*** The lateral faces are triangles, trapezoids, or parallelograms. The volume of a prismatoid whose altitude is h, whose bases are B_1 and B_2, and whose mid-section is M (a *mid-section* is a section halfway between the bases) is given by the formula

$$V = \tfrac{1}{6}h(B_1 + B_2 + 4\,M).$$

(1) Show that in the case of a prism the prismatoid formula becomes $V = Bh$.

(2) Show that in the case of a pyramid the prismatoid formula becomes $V = \frac{1}{3}\,Bh$.

EXERCISES

1. Make a list of the properties of the following solids:
   - (*a*) Regular polyhedrons
   - (*b*) Prisms
   - (*c*) Right prisms
   - (*d*) Parallelepipeds
   - (*e*) Rectangular solids
   - (*f*) Pyramids
   - (*g*) Regular pyramids
   - (*h*) Frustums of pyramids
   - (*i*) Similar polyhedrons

2. Write a formula for each of the following:
   - (*a*) Volume of a rectangular solid
   - (*b*) Volume of a cube
   - (*c*) Volume of a pyramid
   - (*d*) Volume of a frustum of a pyramid
   - (*e*) Total area of a rectangular solid
   - (*f*) Total area of a cube
   - (*g*) Lateral area of a prism
   - (*h*) Lateral area of a regular pyramid

3. Complete the following statements:

(*a*) The volumes of similar polyhedrons are to each other ___?___.

(*b*) The lateral areas of similar polyhedrons are to each other ___?___.

(*c*) If a pyramid is cut by a plane parallel to its base ___?___.

Review Exercises

1. If two planes are perpendicular to a third plane, are they parallel?

2. If a line is perpendicular to a line in a plane, is it perpendicular to the plane?

3. In a right parallelepiped how many faces are necessarily rectangles?

4. Prove: Two planes parallel to a third plane are parallel to each other.

5. Prove: If a plane is passed through the edge of a dihedral angle and perpendicular to the bisecting plane, it makes equal angles with the faces of the dihedral angles.

6. Prove: If perpendiculars to a plane are erected at any number of points of a line in the plane, they all lie in the same plane.

7. The planes of two equilateral triangles ABC and ABC' make a dihedral angle of 60°. If AB is 8 inches, how long is CC'?

8. The planes of two equilateral triangles ABC and ABC' make a dihedral angle of 80°. If AB is 16 inches, how long is CC'? (Use trigonometric tables.)

9. Show how to draw from a point outside a plane three lines which will make equal angles with the plane.

10. Show how to find in a given plane a point which is equally distant from three points not in the plane. Discuss special cases.

11. Through a point draw three lines which are not in the same plane. Show how to pass a plane which will make equal angles with these lines.

12. Through a given point draw a plane which shall be parallel to a given line and perpendicular to a given plane. Prove your construction.

13. What is the locus of a point within a trihedral angle and equally distant from the edges?

14. What is the locus of a point equally distant from the faces of a trihedral angle?

Maintaining Skills[1]

(*Solving equations*)

Solve and check the following equations:

1. $4(x + 2) - 5(x - 2) = 19 - 3(x - 5)$

2. $5(x - 3) + 2(2x - 1) = 6 - (x + 8)$

3. $.02x = .4$

4. $x + 1 = .3x - .75$

5. $(8x - 1)(2x + 3) = (4x - 5)^2 + 3$

6. $(4x + 5)(3x - 2) - (6x + 7)(2x - 5) = 1 - 6x$

7. $\dfrac{2x + 7}{4} - \dfrac{4x + 3}{8} = \dfrac{3x + 8}{5x + 3}$

8. $\dfrac{x - 1}{x - 2} - 2 = \dfrac{x}{1 - x}$

9. $\dfrac{x + a}{x - b} - \dfrac{2(a + b)^2}{x^2 - b^2} = \dfrac{x - a}{x + b}$

Solve and check the simultaneous equations in Exs. 10 and 11:

10. $\begin{aligned} x - 6y &= -10 \\ 2x - 7y &= -15 \end{aligned}$

11. $\begin{aligned} 2x &= y - 1 \\ x - 7 &= -y \end{aligned}$

12. Solve for h: $V = 2\pi r(r + h)$

13. Solve for a, for l, and for r: $S = \dfrac{rl - a}{r - l}$

14. Solve for g: $s = at - \frac{1}{2}gt^2$

15. Solve for t: $h = K\left(l + \dfrac{1}{273}t\right)$

16. Solve for E and for n: $C = \dfrac{nE}{R + nr}$

[1] To the Teacher. See footnote on page 9.

Chapter Test

For each of the following state briefly the relations between the parts of the polyhedrons mentioned:

1. Any two lateral edges of a prism
2. Two parallel sections of a prism
3. A lateral edge and a right section of a prism
4. Any two right sections of a prism
5. A lateral edge and the base of a right prism
6. The altitude and a lateral edge of a right prism
7. The opposite faces of a parallelepiped
8. Any two edges of a cube
9. The volume and an edge of a cube
10. The total area and an edge of a cube
11. The volume and the dimensions of a rectangular solid
12. The area and the dimensions of a rectangular solid
13. The diagonal of a rectangular solid and the three dimensions
14. The edge of a cube and the area of a section made by a plane through two opposite edges
15. The altitude of a pyramid and the base
16. Any two lateral edges of a regular pyramid
17. Any two lateral faces of a regular pyramid
18. The lateral area and an edge of a regular tetrahedron
19. The altitude and an edge of a regular tetrahedron
20. The area of the base of a pyramid and the area of a section parallel to the base made by a plane bisecting the altitude
21. The volume of a pyramid and the volume of a prism with equal base and altitude
22. The volume and an edge of a regular tetrahedron

Miscellaneous Exercises

1. If a cubic foot of ice weighs $57\frac{1}{2}$ pounds, what is the weight of a piece $12 \times 16 \times 20$ inches?

SOLUTION. Volume of the ice, $V = lwh$ (cu. in.).

Volume in cubic feet, $V = \dfrac{lwh}{1728}$.

Weight in pounds, $W = 57\frac{1}{2} \cdot \dfrac{lwh}{1728}$.

$l = 16,\ w = 12,\ h = 20.$

$$W = \frac{115 \cdot \cancel{16}^{1} \cdot \cancel{12}^{1} \cdot \cancel{20}^{10}}{\cancel{2} \cdot \cancel{1728}^{\cancel{144}^{9}}} = \frac{1150}{9} = 127\tfrac{7}{9}.$$

The weight is approximately 130 pounds.

The solution is given here as a model for your numerical work. Note that the formulas to be used are first stated and then the arithmetical work is shown. The answer is rounded off to remind you that answers should not give a false impression of the accuracy in the data.

2. Find the length of each edge of a cube whose total surface is 120 square feet.

3. Each edge of a trirectangular trihedral angle is cut by a plane so that the distance from the vertex of the angle to the intersection of the plane is a units. Find the volume of the pyramid formed.

4. A solid is formed by connecting the center of each face of a cube with the center of the four adjacent faces. If each edge of the cube is 6 inches, what is the volume of this solid?

5. Prove: All the lateral edges of a parallelepiped are equally inclined to the plane of the base.

6. Prove: In a truncated prism no two of whose lateral faces are parallel, not more than one lateral face may be a parallelogram.

7. Prove: All the diagonals of a parallelepiped meet at a point which is the middle point of each diagonal.

8. During a storm $\frac{1}{2}$ inch of rain fell. How many gallons of water fell on a lot 40×190 feet? (A cubic foot of water is equal approximately to 7.5 gallons.)

9. A rectangular tank 10 feet long and 8 feet wide contains 300 gallons of water. How deep is the water?

10. The area of the base of a pyramid is 36 square inches. What is the area of a section of the pyramid parallel to the base and bisecting the altitude?

11. Find the slant height, lateral area, total area, altitude, and volume of a regular tetrahedron whose edge is a units.

12. A plane parallel to the base of a regular quadrangular pyramid bisects the altitude. If the slant height of the pyramid is 12 inches and the lateral edge is 13 inches, what is the volume of the pyramid cut off by the plane? What is the volume of the frustum?

13. A bar of steel 15 feet long has as a cross section a regular hexagon $\frac{3}{4}$ inch on a side. If the steel weighs 499 pounds per cubic foot, what is the weight of the bar?

14. What is the area of a section parallel to the base of a regular triangular pyramid and cutting the altitude 3 units from the vertex if the area of the base is $27\sqrt{3}$ square units and the altitude of the pyramid is 9 units? What part of the whole volume is the volume of the pyramid cut off? What is the total area of the pyramid?

15. If the number of square inches in the total surface of a cube is equal to the number of cubic inches in its volume, how long is its edge?

16. Prove: The volume of a regular prism is equal to one half the product of its lateral area and the apothem of its base.

17. Prove: The lateral area of a pyramid is greater than the area of the base.

18. The water in a swimming pool 60×25 feet is 3 feet deep at one end and 7 feet deep at the other. Assuming that the slope of the floor is constant, find the number of gallons of water in the pool.

19. The edge of a cube is 8 inches. What is the edge of a cube having three times the surface of the first one?

20. Find the volume of a regular tetrahedron whose edge is a units.

21. The area of the base of a parallelepiped is 30 square inches, the length of a lateral edge is 15 inches, and the inclination of the lateral edge to the base is 45°. Find the volume of the parallelepiped.

22. The area of the base of a parallelepiped is 32 square inches, the length of a lateral edge is 20 inches, and the inclination of the lateral edge to the base is 30°. Find the volume of the parallelepiped.

23. How many tons of water will fall on an acre of ground during a rainfall of 1 inch? (A cubic foot of water weighs 62.5 pounds.)

24. Prove: If a section of a tetrahedron made by a plane is a parallelogram, the plane is parallel to a pair of opposite edges.

25. In order to determine the volume of an irregular piece of metal, it was submerged in water in a rectangular container 16 inches long and $10\frac{1}{2}$ inches wide. If the level of the water rose $6\frac{3}{4}$ inches, what was the volume of the metal?

26. The diagonal of a cube is $a\sqrt{3}$ units. What is the volume of the cube?

27. The area of the upper base of a frustum of a pyramid is 3 square inches, and the area of the lower base is 12 square inches. If the volume is 42 cubic inches, what is the altitude?

28. Prove: If three equal lines bisect each other at right angles, their extremities determine a regular octahedron.

29. One edge of a regular octahedron is a units. Find the volume and the total area.

30. The lateral area of a prism is 360 square inches. A right section is an equilateral triangle whose sides are each 18 inches. How long is the lateral edge?

CHAPTER VII

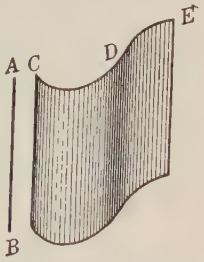
CYLINDERS AND CONES

Cylinders

Ewing Galloway

Cylindrical storage tanks at a petroleum refinery.

§ 199. A *cylindrical surface* is a surface generated by a moving straight line which always intersects a plane curve such as CDE and which is always parallel to a straight line such as AB not in the plane of the curve. When the curve is closed, the surface is called a ***closed cylindrical surface.*** The moving line is called the ***generatrix;*** the curve is the ***directrix.*** The generatrix in any one position is called an ***element*** of the surface.



§ 200. A *cylinder* is a solid bounded by a closed cylindrical surface and two parallel planes cutting all the elements.

§ 201. The parts of the two parallel planes bounded by the cylindrical surface are called the ***bases*** of the cylinder. The part of the cylindrical surface included between the planes is called the ***lateral surface*** of the cylinder.

§ 202. The perpendicular distance between the bases of a cylinder is called the ***altitude.***

§ 203. The ***elements of a cylinder*** are the parts of the elements of the cylindrical surface included between the bases.

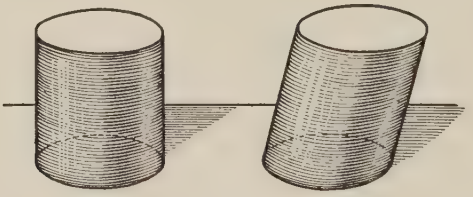
§ 204. COROLLARY. **The elements of a cylinder are parallel and equal.**

§ 205. A ***section*** of a cylinder is the figure formed by the intersection of a plane with the cylinder. If the plane cuts all the elements and is perpendicular to them, the section is a ***right section.***

§ 206. A ***circular cylinder*** is one whose bases are circles. Unless otherwise stated, the word *cylinder* in this book will refer to a circular cylinder. The ***axis*** of a circular cylinder is the line connecting the centers of the bases.

§ 207. A ***right cylinder*** is one in which all the elements are perpendicular to the bases.

§ 208. An ***oblique cylinder*** is one in which all the elements are oblique to the bases.



RIGHT CYLINDER OBLIQUE CYLINDER

§ 209. COROLLARY. **A right circular cylinder may be generated by revolving a rectangle about one of the sides as an axis.**

§ 210. A right circular cylinder is also called a ***cylinder of revolution.*** Explain.

§ 211. COROLLARY. **Through any point on a cylindrical surface there can be but one element.**

(1) Represent a closed cylindrical surface whose directrix is not a circle and whose generatrix makes an angle of 45° with the plane of the directrix.

(2) Represent a right cylinder. Indicate the bases, an element, and the axis.

(3) Represent an oblique cylinder. In it draw a right section.

(4) A rectangle whose dimensions are 3×5 inches revolves about the 3-inch side as an axis. Find the area of one of the bases of the resulting cylinder of revolution.

(5) The area of the base of a cylinder of revolution is 25π square inches, and the altitude is 4 inches. Find the dimensions of a rectangle which would generate this cylinder.

Review Exercises

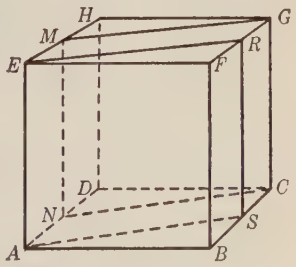
1. Can you tell whether or not two planes are parallel by measuring their distance apart in two places? in three?

2. State the theorem that proves that the corresponding angles of the bases of a prism are equal.

3. A plane passing through two diagonally opposite edges of a parallelepiped forms two triangular prisms. Are these triangular prisms congruent?

4. Prove that the locus of points that are equidistant from A and B and which lie in a plane PQ is a line. (Under what conditions will there be no locus?)

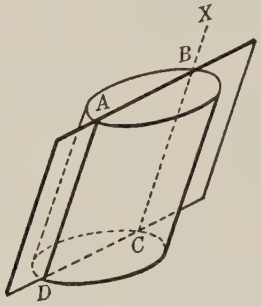
5. *ABCD–EFGH* is a rectangular solid. A plane is passed through *AE* and the middle point *R* of *FG*. Is *CG* parallel to this plane? The plane *MGC* is parallel to plane *AER*. Explain how to construct plane *MGC*. Prove that *ASCN–ERGM* is a right parallelepiped. Prove that the volume of *ASCN–ERGM* is one half the volume of *ABCD–EFGH*.



(Ex. 5)

THEOREM 38

§ 212. Every section of a cylinder made by a plane passing through an element is a parallelogram.



Given the cylinder AC, cut by a plane passing through the element AD and making the section $ABCD$.

To prove that $ABCD$ is a parallelogram.

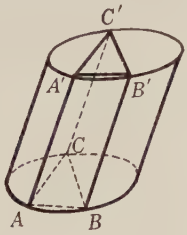
Proof.

STATEMENTS	REASONS
1. Through C in plane DB draw a line $CX \parallel DA$.	1. Syllabus 7.
2. The element through C is also parallel to DA.	2. § 204.
3. CX coincides with the element through C and is therefore in both the plane and the cylindrical surface.	3. Why?
4. CX is the intersection of plane DB and the cylindrical surface.	4. § 17.
5. CB is also the intersection of plane DB and the cylindrical surface.	5. Given.
6. Therefore CB coincides with CX and is a straight line.	6. Why?
7. $CB \parallel DA$.	7. Why?
8. $AB \parallel CD$.	8. Why?
9. $ABCD$ is a parallelogram.	9. Why?

§ 213. COROLLARY. **Every section of a right cylinder made by a plane passing through an element is a rectangle.**

THEOREM 39

§ 214. The bases of a cylinder are congruent.



Given the cylinder BC', with bases ABC and $A'B'C'$.

To prove that base $ABC \cong$ base $A'B'C'$.

Proof. Let A and B be two fixed points and C any other point in the base ABC. AA', BB', and CC' are elements. Prove that $\triangle ABC \cong \triangle A'B'C'$. Place base $A'B'C'$ on base ABC so that $A'B'$ coincides with its equal AB. Then C' will coincide with C. Since C is any point other than A or B, all points of the lower base will coincide with the corresponding points of the upper base. Therefore the bases are congruent.

§ 215. COROLLARY. **Any two parallel sections cutting all the elements of a cylinder are congruent.**

EXERCISES

1. Prove: A line parallel to an element of a cylinder and passing through the center of one base passes through the center of the other base also.

2. Prove: The line joining the centers of the bases of a cylinder is parallel to the elements.

3. Show how to pass a plane through an oblique cylinder so that the section will be a rectangle. Is there more than one such plane?

4. Are all sections of a circular cylinder, not through an element, circles? Discuss all cases.

Inscribed and Circumscribed Prisms

§ **216.** A ***plane tangent to a cylinder*** (Fig. 1) is one which contains an element of a cylinder and does not intersect the cylinder if extended.

Form a cylinder by using a rectangular piece of paper. Place another piece of paper in such a position as to illustrate a tangent plane.

FIG. 1

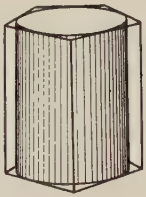


FIG. 2

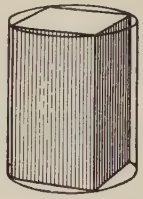


FIG. 3

§ **217.** A ***prism circumscribed about a cylinder*** (Fig. 2) is a prism whose lateral faces are tangent to the cylinder and whose bases are circumscribed about the bases of the cylinder.

§ **218.** A ***prism inscribed in a cylinder*** (Fig. 3) is a prism whose lateral edges coincide with elements of the cylinder and whose bases are inscribed in the bases of the cylinder.

EXERCISES

1. Represent a regular prism inscribed in a cylinder.

2. Represent a regular prism circumscribed about a cylinder.

3. Find the lateral area of a regular triangular prism inscribed in a cylinder with an altitude of 5 inches and a base whose radius is 2 inches.

4. Find the lateral area of a regular triangular prism circumscribed about a cylinder with an altitude of 6 inches and a base whose radius is 3 inches.

5. Find the volumes of regular hexagonal prisms inscribed in and circumscribed about a cylinder with an altitude of 8 inches and a base whose radius is 2 inches.

Area and Volume of a Cylinder

The following exercises will suggest to you a method for finding the area and the volume of a cylinder.

(1) Three prisms whose bases are regular polygons of four, eight, and sixteen sides, respectively, are inscribed in the same cylinder. Show that the lateral area of the octagonal prism is greater than the lateral area of the square prism, and that the lateral area of the prism with sixteen lateral faces is greater than the lateral area of the octagonal prism. Show that the volumes of the three prisms are unequal in the same order.

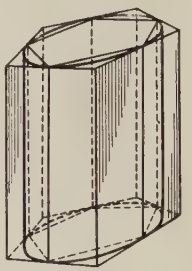
(2) If the process of doubling the number of faces of the prisms as indicated in Exercise 1 is continued indefinitely, will the lateral area and the volume increase without limit?

(3) Three prisms whose bases are regular polygons of four, eight, and sixteen sides, respectively, are circumscribed about the same cylinder. Show that the lateral area of the octagonal prism is less than the lateral area of the square prism, and that the lateral area of the prism with sixteen lateral faces is less than the lateral area of the octagonal prism. Show that the volumes of the three prisms are unequal in the same order.

(4) If the process of doubling the number of faces of the prisms as indicated in Exercise 3 is continued indefinitely, will the lateral area and the volume decrease without limit?

(5) The perimeter of a regular polygon of 512 sides inscribed in a circle of 1-inch diameter is 3.14157 inches; the perimeter of a regular circumscribed polygon of the same number of sides is 3.14163 inches. Find the lateral areas of right prisms having as bases regular polygons of 512 sides which are inscribed in and circumscribed about a cylinder whose diameter is 1 inch and whose altitude is 2 inches. Since the area of the lateral surface of the cylinder lies between the lateral areas of the two prisms, what is the area of the lateral surface of the cylinder to three decimal places?

§ **219.** From a discussion of the exercises on the preceding page, you have seen that the lateral areas and volumes of prisms with regular bases inscribed in a cylinder become greater and greater, and the lateral areas and volumes of such prisms circumscribed about the cylinder become smaller and smaller, as the number of faces is indefinitely increased. The differences between the lateral areas and volumes of the circumscribed prisms and the lateral areas and volumes of the inscribed prisms become less and less. It is obvious, however, that these differences can never be zero. Any desired degree of approximation to the lateral area or volume of a cylinder may be obtained by computing the lateral area or volume of a circumscribed prism or an inscribed prism of a sufficient number of faces.



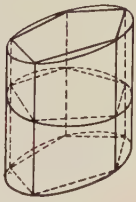
Any property of prisms with regular bases which does not depend upon the number of lateral faces is also a property of circular cylinders.

§ **220.** The ***lateral area of a cylinder*** is the common limit which the lateral areas of the inscribed and circumscribed prisms with regular bases approach when the number of faces is increased indefinitely.

(1) Make a right circular cylinder, using two circles each 3 inches in diameter cut from heavy cardboard for the bases and a rectangular piece of heavy paper for the lateral surface. What must be the dimensions of the rectangle in order that it may be wrapped around the circles without overlapping and make a cylinder 6 inches high? What is the area of this rectangular piece of paper? Is the area of this rectangle the area of the lateral surface of the cylinder? Explain.

§ **221.** The ***volume of a cylinder*** is the common limit which the volumes of the inscribed and circumscribed prisms with regular bases approach when the number of faces is increased indefinitely.

§ 222. Finding the area of a cylinder. (1) What is the lateral area of a prism in terms of an edge and the perimeter of a right section? Does this relationship depend upon the number of lateral faces of the prism? Note that, if a prism with a regular base is inscribed in a cylinder and the number of lateral faces is indefinitely increased, the perimeter of a right section of the prism approaches as a limit the perimeter of a right section of the cylinder. Note also that the lateral edge of the prism remains constantly equal to an element of the cylinder.

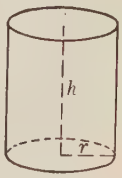


POSTULATE. **The lateral area of a circular cylinder is equal to the product of an element and the perimeter of a right section.**

(2) Find the lateral area and the total area of a cylinder of revolution whose base has a radius 2 inches long and whose altitude is 6 inches.

(3) The radius of the base of a right circular cylinder is r units and the altitude is h units. What is the length of an element? What is the perimeter of a right section? What is the lateral area? What is the total area? Change the expression for the total area to the best form for computation.

§ 223. COROLLARY. **The lateral area L of a cylinder of revolution is equal to the product of its altitude h and the circumference of its base.**



$$L = hC$$
$$\text{or } L = 2\pi rh.$$

§ 224. COROLLARY. **The total area T of a cylinder of revolution whose altitude is h and whose base has a radius r is given by the formula**

$$T = 2\pi r^2 + 2\pi rh = 2\pi r(r + h).$$

EXERCISES

1. From what proposition concerning prisms is the postulate on the preceding page derived?

2. Find the lateral and total areas of right circular cylinders with radii and altitudes respectively as follows:

(*a*) 5 in., 8 in.
(*b*) 3 ft., 10 ft.
(*c*) $3\frac{1}{2}$ in., $9\frac{3}{4}$ in.
(*d*) $\frac{3}{4}$ in., 5 ft.
(*e*) 1.21 in., 5.68 in.
(*f*) 34.2 in., 50.7 in.

3. Solve the formula $L = 2\pi rh$ for r and for h.

4. Solve the formula $T = 2\pi r(r + h)$ for r and for h.

5. If the radius of a cylinder of revolution is 2.1 inches and the lateral area is 18.3 square inches, what is the altitude?

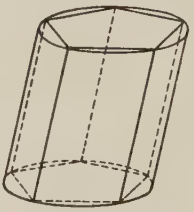
6. If the radius of a cylinder of revolution is 3.4 inches and the total area is 162 square inches, what is the altitude?

7. If the altitude of a right circular cylinder is 5.2 inches and the total area is 315 square inches, what is the radius?

8. Write the formulas $L = 2\pi rh$ and $T = 2\pi r(r + h)$ in terms of h when $h = r$; when $h = 2r$.

9. How much does it cost to paint a cylindrical chimney 30 feet high and 3 feet in diameter at the rate of \$3.25 per 100 square feet?

§ 225. **Finding the volume of a cylinder.** We studied the area of a circular cylinder by comparing it with the area of a regular inscribed or circumscribed prism. Similarly we may study the volume of a circular cylinder by comparing it with the volume of a regular inscribed or circumscribed prism.



(1) What is the volume of a prism in terms of the base and the altitude? Does this relationship depend upon the number of lateral faces of the prism? Note that if a prism with a regular base is inscribed in a cylinder or circumscribed about a cylinder, and the number of lateral faces is indefinitely increased, the area of

the base of the prism approaches as a limit the area of the base of the cylinder. Note also that the altitude of the prism remains constantly equal to the altitude of the cylinder.

POSTULATE. **The volume of a circular cylinder is equal to the product of its base and altitude.**

If V denotes the volume of a circular cylinder, B the area of its base, and h its altitude, then

$$V = Bh.$$

§ 226. COROLLARY. **The volume V of a cylinder of revolution whose altitude is h and whose base has a radius r is given by the formula**

$$V = \pi r^2 h.$$

EXERCISES

1. From what proposition concerning prisms is the postulate in § 225 derived?

2. Find the volumes of cylinders of revolution whose radii and altitudes are, respectively, as follows:

(*a*) 6 in., 10 in. (*c*) $1\frac{1}{4}$ in., 18 ft.
(*b*) 3 in., 15 in. (*d*) .023 in., 3 in.

3. Solve the formula $V = \pi r^2 h$ for h and for r.

4. If the radius of a right circular cylinder is 3.1 inches and the volume is 152 cubic inches, what is the altitude?

5. If the altitude of a cylinder of revolution is 10.2 inches and the volume is 742 cubic inches, what is the radius?

6. Write the formula $V = \pi r^2 h$ in terms of r when $r = h$; then solve for r.

7. An iron pipe whose inside diameter is 1 inch and whose outside diameter is $1\frac{1}{4}$ inches is 10 feet long. If iron weighs 480 pounds per cubic foot, how much does the pipe weigh?

8. In order to find the volume of an irregular piece of ore, it was submerged in a cylindrical tank 8 inches in diameter. The water rose 3 inches. What was the volume of the ore?

9. What is the increase, in gallons, in the amount of water in a circular well 4 feet in diameter when the level of the water rises 2 feet?

Similar Cylinders of Revolution

§ 227. *Similar cylinders of revolution* are cylinders generated by similar rectangles revolving about corresponding sides as axes.

(1) Find the ratio of $2\pi rh$ to $2\pi r'h'$ if $r = 2r'$ and $h = 2h'$.

(2) What is the effect on $2\pi rh$ if h and r are trebled?

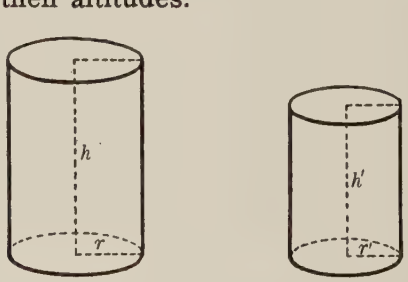
(3) Find the values in terms of π of $2\pi r(r + h)$ and $2\pi r'(r' + h')$ when $r = 3$ inches, $h = 5$ inches, $r' = 6$ inches, and $h' = 10$ inches. Find the ratio of the values.

(4) Prove that, if $\frac{h}{h'} = \frac{r}{r'}$, $\frac{r + h}{r' + h'} = \frac{r}{r'}$.

(5) If $L = 2\pi rh$ and $L' = 2\pi r'h'$, what is the ratio $\frac{L}{L'}$?

Theorem 40

§ 228. The lateral areas or the total areas of similar cylinders of revolution are to each other as the squares of their radii or as the squares of their altitudes; their volumes are to each other as the cubes of their radii or as the cubes of their altitudes.



Given L and L' denoting the lateral areas, T and T' the total areas, V and V' the volumes, h and h' the altitudes, and r and r' the radii of two similar cylinders of revolution.

To prove that

$$\frac{L}{L'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}; \quad \frac{T}{T'} = \frac{r^2}{r'^2} = \frac{h^2}{h'^2}; \quad \frac{V}{V'} = \frac{r^3}{r'^3} = \frac{h^3}{h'^3}.$$

Proof. The proof is left for you.

EXERCISES

1. Find the ratio of the lateral areas and of the volumes of two similar cylinders of revolution whose altitudes have the ratio 2 : 5.

2. If it costs a dollars to paint a cylindrical chimney whose radius is r, what will it cost to paint a similar chimney whose radius is $\frac{3}{4}r$?

3. If water is flowing through a 14-inch pipe at the rate of 600 feet per minute, how many gallons will flow through it during an hour? ($7\frac{1}{2}$ gallons = 1 cubic foot.)

4. Find the number of gallons of gasoline in a cylindrical tank 20 feet long and 6 feet in diameter when it is full.

5. $L = 2\pi rh$ and $T = 2\pi r(r + h)$. Find T in terms of L and h.

6. How high must a cylindrical vessel 10 inches in diameter be if it is to hold 10 gallons? (1 gallon = 231 cubic inches.)

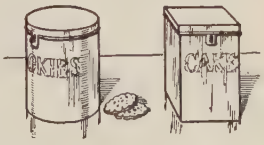
7. If the height of a cylindrical vessel holding a gallon is equal to the diameter of its base, what is the height?

8. If $V = \pi r^2 h$ and $T = 2\pi rh + \pi r^2$, what is the value of T in terms of r and V?

9. By means of higher mathematics it is proved that the largest rectangle which can be inscribed in a circle is one in which the length is equal to the width. Find the volume of the largest prism with a rectangular base which can be inscribed in a right circular cylinder whose altitude is 10 inches and whose radius is 3 inches.

10. When the height of a cylindrical dipper holding a given amount is equal to the radius of the base, the least amount of tin is used. Find the most economical dimensions for a cylindrical tin dipper which is to hold a pint.

11. The heights and lateral surfaces of two tin cans are the same. One is circular; one is square. What is the ratio of their volumes?



(Ex. 11)

Cones

§ 229. A ***conical surface*** is generated by a moving straight line which always intersects a plane curve and always passes through a fixed point not in the plane of the curve. The moving line is called the ***generatrix;*** the curve is the ***directrix;*** the point is the ***vertex.*** The generatrix in any one position is called an ***element*** of the conical surface. When the directrix is a closed curve, the surface is called a ***closed conical surface.***

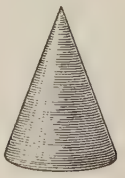
CONICAL SURFACE

§ 230. The parts of the surface on each side of the point are called the ***nappes*** of the conical surface.

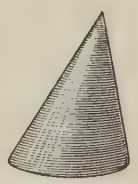
§ 231. A ***cone*** is a solid bounded by one nappe of a closed conical surface and a plane which cuts all the elements. The section formed by the plane is the ***base*** of the cone. The conical surface is the ***lateral surface*** of the cone. The vertex of the conical surface is the ***vertex*** of the cone. The ***elements*** of the cone are the parts of the elements of the conical surface between the vertex and the base.

§ 232. COROLLARY. **Through any point of a conical surface there can be but one element.**

§ 233. A ***circular cone*** is one whose base is a circle. Unless otherwise stated, the word *cone* in this book will refer to a circular cone. The ***axis*** of a circular cone is the line joining the vertex and the center of the base. A ***right circular cone*** is one whose axis is perpendicular to the plane of the base. An ***oblique circular cone*** is one whose axis is oblique to the plane of the base. Any element of a right circular cone is called its ***slant height.***



RIGHT CIRCULAR CONE



OBLIQUE CIRCULAR CONE

§ 234. COROLLARY. **The elements of a right circular cone are equal.**

§ 235. COROLLARY. **A right circular cone may be generated by revolving a right triangle about one of its arms as an axis.**

§ 236. A right circular cone is also called a ***cone of revolution.*** Explain.

§ 237. The ***altitude*** of a cone is the perpendicular distance from the vertex to the plane of the base.

§ 238. COROLLARY. **The altitude and the axis of a right circular cone are the same line.**

§ 239. A ***section*** of a cone is the figure formed by the intersection of a plane with the cone. (See § 21 on page 16.) In a right circular cone a section is a triangle, a circle, an ellipse, an hyperbola, or a parabola, according to the position of the plane. The first two cases are discussed in §§ 240 and 241. The other cases are not considered in this book. In the branch of mathematics called ***analytic geometry***, the properties of conic sections are treated in detail.

EXERCISES

1. Represent an oblique circular cone whose axis makes an angle of 60° with the plane of the directrix.

2. Represent a right circular cone. Indicate the vertex, the base, the altitude, an element, and the axis. Draw a section made by a plane parallel to the base.

3. A right triangle whose sides are 6, 8, and 10 inches revolves about the 8-inch side as an axis. Find the area of the base of the resulting cone of revolution.

4. A right triangle whose sides are 5, 12, and 13 inches revolves about the 13-inch side as an axis. The solid formed is composed of two cones with a common base. Find the area of the common base.

THEOREM 41

§ 240. Every section of a cone made by a plane passing through its vertex is a triangle.

Given a section VAB of a cone made by a plane passing through the vertex V.

To prove that VAB is a triangle.

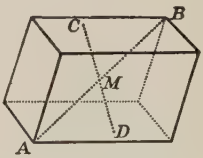
Plan. Do you know at once that VA and VB are straight lines? Is AB a straight line? What must be proved to establish the fact that section VAB is a triangle?

From V draw straight lines to A and B. Show that the straight lines VA and VB are elements of the cone and lines of the given plane. Hence they are the intersection of the plane and the conical surface.

Proof. The proof is left for you.

EXERCISES

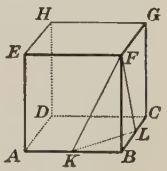
1. Is any straight line drawn through the midpoint of any diagonal of a parallelepiped and terminating in two opposite faces bisected by the point?



(Ex. 1)

SUGGESTION. C and D are in opposite faces of parallelepiped AB; CD passes through midpoint M of diagonal AB. Pass a plane through AB and CD. Is $\triangle ADM \cong \triangle BCM$?

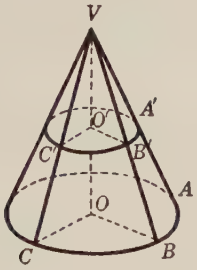
2. AG is a cube. A plane is passed through F and the midpoints K and L of AB and BC. What is the ratio of the volume of F–KBL to the volume of the cube?



(Ex. 2)

THEOREM 42

§ 241. In a circular cone a section made by a plane parallel to the base is a circle.



Given a section $A'B'C'$ of circular cone V–ABC made by a plane parallel to the base ABC.

To prove that $A'B'C'$ is a circle.

Proof.

STATEMENTS	REASONS
1. Draw the axis VO, which intersects the plane $A'B'C'$ at O'.	1. § 6.
2. Through OB, a fixed radius, and VO pass a plane intersecting $A'B'C'$ in $O'B'$ and the conical surface in the straight line VB.	2. §§ 13, 19, 240.
3. Through OC, any other radius, and VO pass a plane intersecting $A'B'C'$ in $O'C'$ and the conical surface in VC.	3. §§ 13, 19, 240.
4. $O'B' \parallel OB$, $O'C' \parallel OC$.	4. Why?
5. $\triangle VBO \sim \triangle VB'O'$, $\triangle VCO \sim \triangle VC'O'$.	5. Why?
6. $\frac{OB}{O'B'} = \frac{VO}{VO'}$, $\frac{OC}{O'C'} = \frac{VO}{VO'}$.	6. Why?
7. $OB = OC$.	7. Why?
8. $O'B' = O'C'$.	8. Syllabus 121.
9. $A'B'C'$ is a circle.	9. Why?

(1) VO and VO' are 12 inches and 3 inches, respectively. What is the ratio of $A'B'C'$ to ABC?

§ 242. COROLLARY. **The axis of a circular cone passes through the centers of all sections parallel to the base.**

§ 243. COROLLARY. **The area of a section made by a plane parallel to the base of a circular cone is to the area of the base as the square of its distance from the vertex of the cone is to the square of the altitude.**

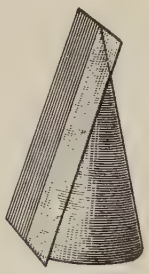
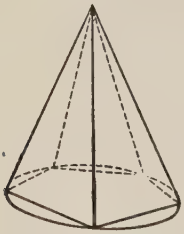


FIG. 1

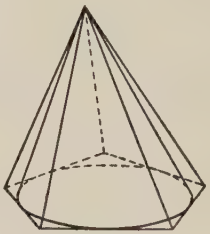
§ 244. A plane is said to be ***tangent to a cone*** if it passes through an element but contains no other point of the cone (Fig. 1).

§ 245. COROLLARY. **A plane tangent to a cone is determined by a tangent to the base of the cone and an element drawn to the point of contact.**

§ 246. A pyramid is said to be ***inscribed in a cone*** when its vertex coincides with the vertex of the cone and its base is inscribed in the base of the cone.



INSCRIBED PYRAMID



CIRCUMSCRIBED PYRAMID

§ 247. COROLLARY. **The lateral edges of a pyramid inscribed in a cone are elements of the cone.**

§ 248. A pyramid is said to be ***circumscribed about a cone*** when its vertex coincides with that of the cone and its base is circumscribed about the base of the cone.

§ 249. COROLLARY. **The faces of a pyramid circumscribed about a cone are tangent to the cone.**

EXERCISES

1. What kind of triangle is any section made by a plane passed through the axis of a right circular cone?

2. The slant height of a right circular cone is 13 inches. The radius of the base is 5 inches. Find the length of the altitude.

3. If the altitude of a right circular cone is 8 inches and the slant height is 10 inches, what is the area of the base?

4. The radius of the base of a right circular cone is r units and the slant height is s units. Find the length of the altitude.

5. The altitude of a circular cone is 10 inches and the radius of the base is 3 inches. Find the area of a section parallel to the base and 4 inches from the vertex.

6. A plane is passed perpendicular to the altitude of a right circular cone at its mid-point. What is the ratio of the area of the section to the area of the base?

7. Represent a regular pyramid inscribed in a cone.

8. Represent a regular pyramid circumscribed about a cone.

9. Find the lateral area of a regular triangular pyramid inscribed in a cone with an altitude of 6 inches and a base whose radius is $2\frac{1}{2}$ inches.

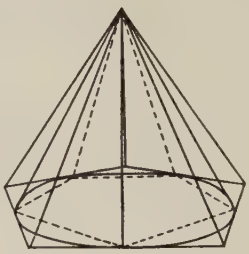
10. Find the volume of a regular triangular pyramid circumscribed about a cone with an altitude of 8 inches and a base whose radius is 3 inches.

11. Find the lateral area of a regular quadrangular pyramid inscribed in a cone with an altitude of 5 inches and a base whose radius is $1\frac{1}{2}$ inches.

12. Find the volume of a regular hexagonal pyramid circumscribed about a cone with an altitude of 10 inches and a base whose radius is 4 inches.

Area and Volume of a Cone

§ **250.** Properties of cones are derived from properties of pyramids in the same way that properties of cylinders are derived from properties of prisms. The lateral areas and volumes of pyramids with regular bases inscribed in a cone become greater and greater, and the lateral areas and volumes of such pyramids circumscribed about the cone become smaller and smaller, as the number of faces is indefinitely increased. The differences between the lateral areas and volumes of the circumscribed pyramids and the lateral areas and volumes of the inscribed pyramids become less and less, but they can never be zero.



Any desired degree of approximation to the lateral area or volume of a cone may be obtained by computing the lateral area or volume of a circumscribed or an inscribed pyramid of a sufficient number of faces.

Any property of pyramids with regular bases which does not depend upon the number of lateral faces is also a property of circular cones.

§ **251.** The ***lateral area of a cone*** is the common limit which the lateral areas of the inscribed and circumscribed pyramids with regular bases approach when the number of faces is increased indefinitely.

§ **252.** The ***volume of a cone*** is the common limit which the volumes of the inscribed and circumscribed pyramids with regular bases approach when the number of faces is increased indefinitely.

§ **253. Finding the area of a cone.** (1) A regular octagon and a square are circumscribed about the same circle. Prove that the perimeter of the octagon is less than the perimeter of the square.

(2) What is the lateral area of a regular pyramid in terms of the slant height and perimeter of the base? Does this relationship depend upon the number of lateral faces of the pyramid? Note that if a regular pyramid is circumscribed about a cone of revolution and the number of lateral faces is indefinitely increased, the perimeter of the base of the pyramid approaches as a limit the circumference of the base of the cone. Note also that the slant height of the pyramid is constantly equal to the slant height of the cone.

POSTULATE. **The lateral area of a cone of revolution is equal to half the product of its slant height and the circumference of its base.**

If L denotes the lateral area of the cone of revolution, s its slant height, and C the circumference of its base, then

$$L = \tfrac{1}{2}\, sC.$$

(1) From what proposition concerning pyramids is this postulate derived?

(2) The slant height of a cone of revolution is 5 inches and the radius of the base is 3 inches. Find the lateral area.

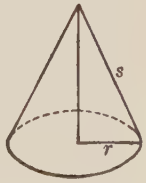
(3) Find the total area of the cone in Exercise 2.

(4) The altitude of a cone of revolution is 6 inches and the radius of the base is 4 inches. Find the lateral area.

(5) Find the total area of the cone in Exercise 4.

(6) The slant height of a right circular cone is s and the radius of the base is r. What is the lateral area? Find the total area, and change your answer to the best form for computation.

§ 254. COROLLARY. **If the radius of the base of a cone of revolution is denoted by r and the slant height is denoted by s, the lateral area L and the total area T are given by the following formulas:**



$$L = \pi rs$$
$$T = \pi r^2 + \pi rs = \pi r(r + s).$$

EXERCISES

1. Find the lateral areas and the total areas of right circular cones with dimensions as follows:

(*a*) Radius of the base 6 inches, slant height 15 inches
(*b*) Radius of the base 8.2 inches, slant height 16 inches
(*c*) Radius of the base 6 inches, altitude 8 inches
(*d*) Radius of the base 8 inches, altitude 12 inches
(*e*) Slant height 10.3 inches, altitude 6.2 inches
(*f*) Radius of the base a inches, altitude b inches

2. Solve the formula $L = \pi rs$ for r and for s.

3. Solve the formula $T = \pi r(r + s)$ for r and for s.

4. If the radius of the base of a cone of revolution is 3.2 inches and the lateral area is 40.4 square inches, what is the slant height?

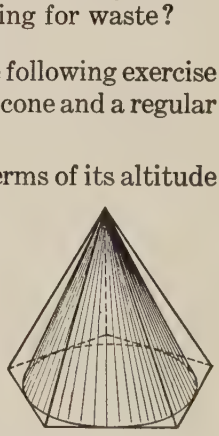
5. If the slant height of a cone of revolution is 4.7 inches and the total area is 75.3 square inches, what is the radius of the base?

6. Write the formulas $L = \pi rs$ and $T = \pi r(r + s)$ in terms of r when $s = 2r$.

7. If the slant height of a conical tent is 12 feet and the diameter of the base is 14 feet, how many square yards of canvas were required to make it, not allowing for waste?

§ 255. **Finding the volume of a cone.** The following exercise will indicate certain relationships between a cone and a regular pyramid circumscribed about the cone.

(1) What is the volume of a pyramid in terms of its altitude and the area of its base? Does this relationship depend upon the number of lateral faces of the pyramid? Note that if a pyramid with a regular base is circumscribed about a cone and the number of lateral faces is indefinitely increased, the area of the base of the pyramid approaches as a limit the area of the base of the cone. (Visualize tangents drawn at the middle points of the arcs,



then tangents at the middle points of the resulting arcs, and so on indefinitely.) Note also that the altitude of the pyramid remains constantly equal to the altitude of the cone.

POSTULATE. **The volume of a circular cone is equal to one third the product of its base and altitude.**

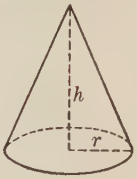
If B denotes the area of the base of a circular cone, h its altitude, and V its volume, then

$$V = \tfrac{1}{3} Bh.$$

(1) From what proposition concerning pyramids is the above postulate derived?

(2) Find the volume of a circular cone whose base has a radius 3 inches long and whose altitude is 7 inches.

(3) Find the volume of a circular cone whose base has a radius r units long and whose altitude is h units.



§ 256. COROLLARY. **The volume V of a circular cone whose altitude is h and whose base has a radius r is given by the formula**

$$V = \tfrac{1}{3} \pi r^2 h.$$

EXERCISES

1. Find the volumes of circular cones with altitudes and areas of bases respectively as follows:

(*a*) 3 in., 8 sq. in.
(*b*) $3\frac{1}{4}$ in., $8\frac{5}{8}$ sq. in.
(*c*) $4\frac{3}{8}$ in., $9\frac{1}{2}$ sq. in.
(*d*) 9.4 in., 18.6 sq. in.

2. Find the volumes of cones with altitudes and radii of bases respectively as follows:

(*a*) 12 in., 5 in.
(*b*) 7 in., $3\frac{1}{2}$ in.
(*c*) 8.9 in., 3.4 in.
(*d*) 29.4 in., 11.6 in.

3. Find the volume of a cone of revolution whose altitude is 6 inches and whose slant height is 9 inches.

4. Find the volume of a right circular cone whose altitude is h units and whose slant height is s units.

5. Solve the formula $V = \frac{1}{3}\pi r^2 h$ for h and for r.

6. What does the formula $V = \frac{1}{3}\pi r^2 h$ become in terms of r when $r = h$?

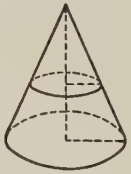
7. If the radius of the base of a circular cone is 2.4 inches and the volume is 70.2 cubic inches, what is the altitude?

8. Find the slant height of a cone of revolution whose volume is $30\,\pi$ cubic inches and whose altitude is 10 inches.

9. Find the volume of a right circular cone whose altitude is a units and whose slant height is $2\,a$ units.

10. A pile of grain has the shape of a cone 3 feet high. The circumference of the base is $12\frac{1}{2}$ feet. How many bushels of grain are in the pile? (A bushel is 2150.42 cubic inches or approximately $1\frac{1}{4}$ cubic feet.)

11. A right circular cone whose altitude is 10 inches is cut by a plane parallel to the base and 6 inches from the vertex. If the radius of the base of the cone is 5 inches, what are the lateral areas and the volumes of the two parts into which the cone is cut?



(Ex. 11)

Similar Cones

§ **257.** ***Similar cones of revolution*** are cones generated by similar right triangles revolving about corresponding arms as axes.

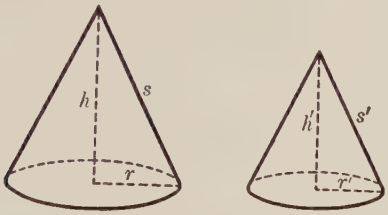
(1) What is the value of the ratio $\frac{\pi r s}{\pi r' s'}$ in terms of r and r' if $\frac{r}{r'} = \frac{s}{s'}$?

(2) What is the value of the ratio $\frac{\frac{1}{3}\pi r^2 h}{\frac{1}{3}\pi r'^2 h'}$ in terms of r and r' if $\frac{r}{r'} = \frac{h}{h'}$?

(3) What is the value of the ratio $\frac{\pi r\,(r+s)}{\pi r'(r'+s')}$ in terms of r and r' if $\frac{r}{r'} = \frac{s}{s'}$?

THEOREM 43

§ 258. The lateral areas or the total areas of similar cones of revolution are to each other as the squares of their altitudes, or the squares of their slant heights, or the squares of the radii of their bases; their volumes are to each other as the cubes of their altitudes, or the cubes of their slant heights, or the cubes of the radii of their bases.



Given L and L' denoting the lateral areas, T and T' the total areas, V and V' the volumes, h and h' the altitudes, s and s' the slant heights, and r and r' the radii of the bases of similar cones of revolution.

To prove that

$$(1)\quad \frac{L}{L'} = \frac{h^2}{h'^2} = \frac{s^2}{s'^2} = \frac{r^2}{r'^2}.$$

$$(2)\quad \frac{T}{T'} = \frac{h^2}{h'^2} = \frac{s^2}{s'^2} = \frac{r^2}{r'^2}.$$

$$(3)\quad \frac{V}{V'} = \frac{h^3}{h'^3} = \frac{s^3}{s'^3} = \frac{r^3}{r'^3}.$$

The proofs of (1) and (3) are left for you. We give here the plan of the proof of (2).

Plan of proof. Since $T = \pi r(r + s)$ and $T' = \pi r'(r' + s')$,

$$\frac{T}{T'} = \frac{r(r + s)}{r'(r' + s')} = \frac{r}{r'} \times \frac{r + s}{r' + s'}.$$

But in two similar cones of revolution $\frac{r}{r'} = \frac{s}{s'}$.

Hence $\frac{r + s}{r' + s'} = \frac{r}{r'}$ and $\frac{r}{r'} \times \frac{r + s}{r' + s'} = \frac{r^2}{r'^2}$.

EXERCISES

1. What will be the effect upon the total area of a cone of revolution if all its dimensions are doubled?

2. What will be the effect upon the volume of a cone of revolution if all its dimensions are increased in the ratio 12 : 5?

3. What is the ratio of the altitudes of two similar cones of revolution whose lateral areas have the ratio 1 : 3?

4. What is the ratio of the altitudes of two similar cones of revolution whose volumes have the ratio 1 : 8?

Frustums of Cones

§ 259. A *frustum of a cone* is a part of a cone included between the base and a plane parallel to the base (Fig. 1). The base of the cone and the section parallel to the base are called the ***bases of the frustum.*** The curved surface is called the ***lateral surface of the frustum.***

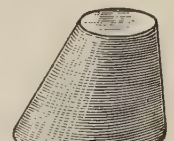


FIG. 1

§ 260. The ***altitude of a frustum*** of a cone is the perpendicular distance between the bases.

(1) Represent a frustum of a cone and its altitude.

§ 261. The ***slant height of a frustum of a right circular cone*** is that part of the slant height of the cone which is included between the bases of the frustum.

A discussion similar to that in § 250 leads to the following definitions and postulates:

§ 262. The ***lateral area of a frustum of a cone*** is the common limit which the lateral areas of the inscribed and circumscribed frustums of pyramids with regular bases approach when the number of lateral faces is increased indefinitely.

§ 263. The ***volume of a frustum of a cone*** is the common limit which the volumes of the inscribed and circumscribed frustums of pyramids with regular bases approach when the number of lateral faces is increased indefinitely.

§ 264. POSTULATE. The lateral area of a frustum of a cone of revolution is equal to half the sum of the circumferences of the bases multiplied by the slant height of the frustum.

If L denotes the lateral area, c and c' the circumferences of the bases, and s the slant height, then

$$L = \tfrac{1}{2}s(c + c').$$

§ 265. POSTULATE. The volume V of a frustum of a circular cone the areas of whose bases are B and B' and whose altitude is h is given by the formula

$$V = \tfrac{1}{3}h(B + B' + \sqrt{BB'}).$$

(1) If the radii of the bases of a frustum of a right circular cone are 2 inches and 5 inches and the slant height is 3 inches, what is the lateral area?

(2) If the radii of the bases of a frustum of a cone of revolution are r and r' and the slant height is s, what is the lateral area? Simplify the result.

(3) If the radii of the bases of a frustum of a circular cone are 2 inches and 3 inches and the altitude is 4 inches, what is the volume?

(4) If the radii of the bases of a frustum of a circular cone are r and r' and the altitude is h, what is the volume? Simplify the result.

§ 266. COROLLARY. If the radii of the bases of a frustum of a cone of revolution are r and r' and the slant height is s, the lateral area L of the frustum is given by the formula

$$L = \pi s(r + r').$$

§ 267. COROLLARY. If the radii of the bases of a frustum of a circular cone are r and r' and the altitude is h, the volume V of the frustum is given by the formula

$$V = \tfrac{1}{3}\pi h(r^2 + r'^2 + rr').$$

EXERCISES

1. Find the lateral areas of frustums of cones of revolution whose slant heights and radii of bases are respectively as follows:

(*a*) 4 in., 5 in., 3 in.
(*b*) 6 in., $4\frac{1}{2}$ in., $3\frac{1}{4}$ in.
(*c*) 7.5 in., 4.6 in., 3.1 in.
(*d*) 6.2 ft., 3.7 ft., 2.2 ft.

2. Find the volumes of frustums of circular cones whose altitudes and radii of bases are respectively as follows:

(*a*) 10 in., 7 in., 5 in.
(*b*) 5.7 in., 3.2 in., 2.3 in.
(*c*) 7.5 in., 4.7 in., 3.2 in.
(*d*) $4\frac{1}{2}$ ft., $3\frac{1}{3}$ ft., $1\frac{3}{4}$ ft.

3. Show that the lateral area of a frustum of a cone of revolution is equal to the circumference of a section midway between its bases multiplied by its slant height.

4. Solve the formula $L = \pi s(r + r')$ for s and for r.

5. Solve the formula $V = \frac{1}{3}\pi h(r^2 + r'^2 + rr')$ for h.

6. The masonry of a Dutch windmill 40 feet high is in the form of a hollow frustum of a right circular cone. The outer diameters of the top and the bottom are 40 feet and 52 feet, respectively. The inner diameters are 33 feet and 40 feet. How many cubic feet of masonry are there in the windmill?

7. Make a list of the properties of the following solids:

(*a*) Cylinders
(*b*) Cones
(*c*) Frustums of cones of revolution
(*d*) Similar cylinders
(*e*) Cylinders of revolution
(*f*) Cones of revolution
(*g*) Similar cones

8. Write a formula for each of the following:

(*a*) The lateral area of a cylinder of revolution
(*b*) The total area of a cylinder of revolution
(*c*) The volume of a circular cylinder
(*d*) The volume of a cylinder of revolution
(*e*) The lateral area of a cone of revolution
(*f*) The total area of a cone of revolution
(*g*) The volume of a circular cone
(*h*) The lateral area of a frustum of a cone of revolution
(*i*) The volume of a frustum of a circular cone

Chapter Test

Complete the statements in Exercises 1–21.

1. The elements of a cylinder are ___?___ and ___?___.

2. The elements of a ___?___ ___?___ cone are equal.

3. A right cylinder is one in which all the ___?___ are ___?___ to the ___?___.

4. Every section of a cylinder made by a plane passing through an element is a ___?___.

5. The bases of a cylinder are ___?___.

6. The faces of a pyramid circumscribed about a cone are ___?___ ___?___ the cone.

7. Any element of a right circular cone is called its ___?___ ___?___.

8. A right circular cone is also called a ___?___ ___?___ ___?___.

9. Every section of a cone made by a plane passing through its vertex is a ___?___.

10. In a circular cone a section made by a plane parallel to the base is a ___?___.

11. The total area of a cylinder of revolution whose altitude is h inches and whose base has a radius of r inches is ___?___ square inches.

12. The volume of a cylinder of revolution whose altitude is h inches and whose base has a radius of r inches is ___?___ cubic inches.

13. If the altitude of a cylinder of revolution is $2\,a$ inches and the radius is $\frac{3\,a}{2}$ inches, the volume is ___?___ cubic inches.

14. The formula for the volume of a right circular cone whose altitude is h and the radius of whose base is r is ___?___.

15. If the radius of the base of a cone of revolution is $3\,a$ inches and the slant height is $5\,a$ inches, the total area is ___?___ square inches.

16. If the radius of the base of a cone of revolution is r inches and the slant height is s inches, the altitude is $\frac{?}{}$ inches.

17. If the altitude of a right circular cylinder is trebled, the radius remaining the same, the lateral area is increased in the ratio $\frac{?}{}$ and the volume is increased in the ratio $\frac{?}{}$.

18. If the radius of a cone of revolution is doubled, the altitude remaining the same, the lateral area is increased in the ratio $\frac{?}{}$ and the volume is increased in the ratio $\frac{?}{}$.

19. If both the altitude and the radius of a cylinder of revolution are doubled, the volume is increased in the ratio $\frac{?}{}$.

20. If from a cone of revolution whose altitude is $2\,a$ inches and the radius of whose base is $2\,r$ inches a cone whose altitude is a inches is cut off, the volume of the remaining frustum is $\frac{7\,?}{3}$ cubic inches.

21. The volume of a regular hexagonal prism inscribed in a cylinder with an altitude of h inches and a base whose radius is r inches is $\frac{?\sqrt{3}}{2}$ cubic inches.

22. Through one vertex of an equilateral triangle whose sides are each 8 inches draw a line in the plane of the triangle perpendicular to the altitude from that vertex. If the triangle rotates about the perpendicular as an axis, what are the area and the volume of the generated solid?

23. The volume of a right circular cylinder of diameter 3 inches is 30 cubic inches. What is the volume of a similar cylinder of diameter 6 inches? Find the lateral areas of each of these cylinders.

24. A solid wooden cone of revolution is divided into two parts by a cut parallel to the base and halfway between the base and the vertex. What is the ratio of the weights of the two parts?

Miscellaneous Exercises

1. What is the locus of points in space 5 inches from a given line?

2. What is the locus of points in space d units from a given line?

3. Find the lateral area of a right circular cylinder whose diameter is 1 inch and whose altitude is 2 inches. Find the lateral areas of the regular circumscribed and regular inscribed hexagonal prisms.

4. How many square yards of canvas will be required to make a conical tent 8 feet high and 12 feet in diameter at the base, not allowing for waste or overlapping?

5. Find the altitude of a cylinder whose diameter is 5 inches and whose volume is equal to that of a cube 6 inches on each edge.

6. What is the volume of a frustum of a right circular cone if the radii of the bases are 3 inches and 8 inches and the altitude is 6 inches? What is the lateral area?

7. Two lines each 8 inches long meeting at an angle of 60° bisect each other. If they are revolved about the bisector of the 60° angle as an axis, what is the area of the surface generated?

8. What is the volume of a circular cylinder whose radius is 3 inches and whose element is 6 inches, if the inclination of an element to the base is 45°? 60°? 30°?

9. A tin vessel, open at the top, is in the shape of a frustum of a cone. If the diameters at the top and the bottom are 8 inches and 5 inches respectively and it is 7 inches high, how much tin was required to make it and what is its volume?

10. If the lateral area of a right circular cylinder whose altitude is 10 inches is 200 square inches, what is the radius of the base?

11. What is the nature of the surface formed by all the straight lines which make a given angle with a given line at a given point?

12. Find the lateral area of a right circular cylinder whose radius is 3 inches and whose altitude is 5 inches. Find also the lateral areas of the regular inscribed and circumscribed triangular prisms.

13. What is the nature of the surface formed by all the straight lines which make a given angle with a given plane at a point in the plane?

14. A moving line AB, 8 inches long, meets a plane MN at a fixed point A, always making with the plane an angle of 60°. What is the area of the surface generated by AB?

15. The volume of a cone is 300 cubic inches. The slant height is equal to the diameter of the base. Find the altitude.

16. An isosceles trapezoid whose bases are 10 inches and 6 inches and whose altitude is 5 inches is revolved about its longer base. What are the volume and the area of the solid generated?

17. A pail is 11 inches in diameter at the top and $8\frac{1}{2}$ inches in diameter at the bottom. It is $9\frac{3}{4}$ inches high. Will it hold 12 quarts as advertised?

18. A boiler contains 120 tubes, each $3\frac{7}{8}$ inches in outside diameter and 16 feet long. What is the lateral area of all the tubes to the nearest square foot?

19. Find the ratio of the volume of a regular triangular prism to the volume of the circumscribed cylinder.

20. Find the ratio of the volume of a regular hexagonal prism to the volume of the inscribed cylinder.

21. $L = 2\pi rh$ and $\pi r^2 h = 300$. Find L in terms of r.

22. Write a formula for finding the lateral area of a cone of revolution in terms of the radius of the base and the altitude.

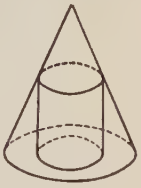
23. Write a formula for finding the volume of a cone of revolution in terms of the slant height and the radius of the base.

24. Write a formula for the total area of a cone of revolution in terms of the radius of the base and the altitude.

25. A right circular cylinder 18 inches in diameter and 20 inches high is surmounted by a right circular cone of the same diameter and 10 inches high. Find the lateral area of the resulting solid.

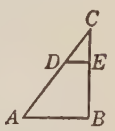
26. For a cylindrical tin can which holds a quart, a minimum amount of tin is used when the height is equal to the diameter. Find the dimensions of such a can.

27. A cylindrical tank 8 feet long and 3 feet in diameter is placed so that its axis is horizontal. It is filled with oil to a depth of 9 inches measured along a vertical radius. How many gallons of oil does it contain?



(Ex. 28)

28. The altitude of the cylinder of greatest lateral surface which can be inscribed in a given cone of revolution is one half the altitude of the cone. If the altitude of the cone is 12 inches and the diameter of the base is 10 inches, what are the dimensions of the cylinder? If the altitude of the cone is h and the diameter of the base is $2r$, what are the dimensions of the cylinder?



(Ex. 29)

29. In the right triangle ABC, $AB = 6$ inches, $BC = 8$ inches, and $\angle B = 90°$. DE is drawn parallel to AB so that $CE = 3$ inches. If the figure is revolved about BC as an axis, what is the lateral area of that part generated by $ABED$?

30. Find the ratio of the volumes of a regular triangular pyramid and the circumscribed cone.

31. Find the ratio of the volume of a regular hexagonal pyramid to the volume of the inscribed cone.

32. If the altitude of a cone is doubled and the radius of the base remains the same, what is the effect upon the volume?

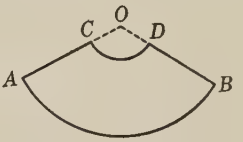
33. If the radius of the base of a cone is doubled and the altitude remains the same, what is the effect upon the volume? What is the effect upon the lateral area?

34. If a cone and a cylinder have the same base and equal volumes, what is the ratio of the altitudes?

35. At what point of the axis of a cone of revolution should a plane parallel to the base be passed in order to cut off a cone equal in volume to one half the given cone?

36. What must be the radius of a cylindrical tin can which will hold the same amount as two cans whose diameters are 3 inches and 4 inches respectively, the height of all three cans being the same?

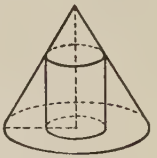
37. If a model of a cone is made from a semicircle whose radius is 3 inches, what are the lateral area and the volume of the cone?



(Ex. 38)

38. A model of a frustum of a cone may be made by using a piece of paper cut as shown. AB is an arc of a circle whose radius is 10 inches. CD is an arc of a circle whose radius is 3 inches. The angle at O is 120°. What is the lateral area of the frustum?

39. A right circular cylinder fits in a right circular cone as shown in the figure. The altitude of the cone is 10 inches, and the radius of its base is 6 inches. If the altitude of the cylinder is 6 inches, what is the radius of its base?


(Ex. 39)

40. Two similar cones of revolution are generated by similar right triangles whose hypotenuses are 10 inches and 15 inches respectively. What are the ratios of the total areas and of the volumes?

41. Prove: A plane which is tangent to a right circular cylinder is perpendicular to the planes of the bases.

42. Prove: The intersection of two planes tangent to the same cone passes through the vertex of the cone.

43. Show how to pass a plane through the vertex of an oblique circular cone in order to obtain a section which is an isosceles triangle.

44. Two planes tangent to a cone of revolution pass through the extremities of a diameter of the base. Prove that the intersection is parallel to the plane of the base.

CHAPTER VIII

SPHERES

Definitions

§ 268. A *sphere* is a surface all points of which are equally distant from a point called the *center*.[1] (Fig. 1.) Compare this definition with that of a circle.

FIG. 1

If you look at the picture of the Hayden Planetarium in the front of this book, you will notice that the shadows cast by the spherical dome of the Planetarium are circular in shape. Aristotle, the Greek philosopher, observed that during an eclipse the earth cast a circular shadow on the moon, and he is believed to have been the first to use this fact as evidence that the earth was spherical.

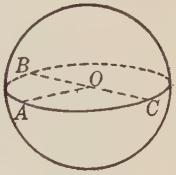


FIG. 2

§ 269. A *radius of a sphere* is a straight line drawn from the center to any point of the sphere. In Fig. 2, OA is a radius.

§ 270. A *diameter of a sphere* is a line through the center with its extremities on the sphere. In Fig. 2, BC is a diameter.

§ 271. A *chord of a sphere* is a straight line connecting any two points on the sphere.

§ 272. POSTULATE. **One and only one sphere may be constructed with a given center and a given radius.**

§ 273. COROLLARY. **A diameter of a sphere is twice the radius.**

§ 274. COROLLARY. **Two spheres are equal if their radii or their diameters are equal.**

[1] As this definition indicates, the word *sphere* refers to the surface; but where no ambiguity arises, it may be used to refer either to the surface or the space enclosed by the surface.

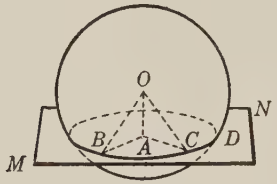
§ 275. COROLLARY. **All radii and all diameters of the same sphere or of equal spheres are equal.**

§ 276. COROLLARY. **A sphere may be generated by revolving a semicircle about its diameter as an axis.**

§ 277. COROLLARY. **A point is within, on, or without a sphere according as its distance from the center is less than, equal to, or greater than a radius.**

THEOREM 44

§ 278. Every section of a sphere made by a plane is a circle.



Given a section BCD of a sphere O made by a plane MN. O is the center of the sphere.

To prove that BCD is a circle.

Plan. Draw OA perpendicular to MN, meeting MN at A. Take C, a fixed point on the section, and B, any other point on the section. Prove that AB equals AC. Therefore every point on the section is at the same distance from A.

Proof. The proof is left for you.

§ 279. COROLLARY. **A line through the center of a sphere and perpendicular to an intersecting plane meets the plane at the center of the circle of intersection.**

§ 280. COROLLARY. **A line connecting the center of a sphere with the center of any section of the sphere made by a plane is perpendicular to the plane.**

Circles of a Sphere

(1) In a sphere whose radius is 5 inches, find the area of a section made by a plane which is 3 inches from the center.

(2) In a sphere whose radius is a units, how many sections are there made by planes b units ($a > b$) from the center? Show that these sections are all equal.

(3) In a sphere whose radius is 10 inches, sections are made by planes 4 inches and 6 inches from the center. Show that the circle made by the plane 4 inches from the center is larger than the circle made by the plane 6 inches from the center.

(4) In a sphere whose radius is a units, sections are made by planes b and c units from the center ($a > b > c$). Show that the circle made by the plane b units from the center is less than the circle made by the plane c units from the center.

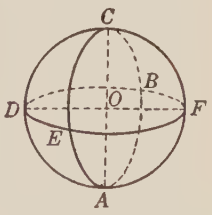
§ 281. COROLLARY. **Circles of a sphere made by planes equally distant from the center are equal.**

§ 282. COROLLARY. **The planes of equal circles on a sphere are equally distant from the center of the sphere.**

§ 283. COROLLARY. **Of two circles on a sphere made by planes not equally distant from the center the one made by the plane nearer the center is the greater.**

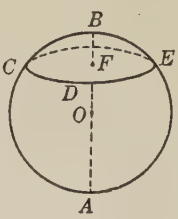
§ 284. COROLLARY. **The planes of unequal circles on a sphere are unequally distant from the center, the greater circle being nearer the center.**

§ 285. A ***great circle of a sphere*** is a section made by a plane passing through the center. (The circles ABC and DEF are great circles of sphere O.) All other sections are called ***small circles of the sphere.*** In the figure at the top of the next page, CDE is a small circle of sphere O.



§ 286. The ***axis of a circle of a sphere*** is the diameter of the sphere which is perpendicular to the plane of the circle.

The ends of the axis are called the *poles* of the circle. Thus AB is the axis of circle F. A and B are the poles of the circle.



§ 287. As you have seen, a drawing of a sphere usually represents a great circle and one or more other circles of the sphere.

(1) Represent the earth as a sphere and on it place circles representing the equator, the tropic of Cancer, the tropic of Capricorn, and a great circle through the north and south poles.

(2) Of what great circle on the earth is the north pole the pole?

(3) Are the parallels of latitude on the earth parallel lines? Why are they called parallels?

(4) Show that the center of a sphere is the center of any great circle of that sphere.

(5) In a sphere whose radius is 10 inches, what is the diameter of any great circle?

(6) Represent a small circle of a sphere and its axis. Where are the poles of this circle? Where are the poles of any circle which is parallel to this circle?

(7) Represent a great circle of a sphere, its axis, and its poles. Pass a plane through the center of the sphere perpendicular to the plane of the great circle. Does this plane contain the poles of the great circle? Explain.

(8) Prove: The intersection of the planes of two great circles is a diameter of the sphere.

(9) May a circle of a sphere always be drawn through three points on a sphere? How many circles are there through any three points on a sphere? Explain.

(10) How many great circles may be drawn through two points which are at the ends of a diameter of a sphere?

(11) If two points on a sphere are not at the ends of a diameter, how many great circles may be drawn through these two points? How many small circles? Explain.

§ 288. COROLLARY. **The axis of a circle of a sphere passes through the center of the circle.**

§ 289. COROLLARY. **A diameter of a sphere passing through the center of a small circle is the axis of the circle.**

§ 290. COROLLARY. **All great circles of the same sphere are equal.**

§ 291. COROLLARY. **Circles of the same sphere in parallel planes have the same axis and the same poles.**

§ 292. COROLLARY. **Every great circle bisects the sphere.**

§ 293. COROLLARY. **Any two great circles of a sphere bisect each other.**

§ 294. COROLLARY. **If the planes of two great circles are perpendicular to each other, each circle passes through the poles of the other.**

§ 295. COROLLARY. **One and only one circle may be drawn through any three points on the surface of a sphere.**

§ 296. COROLLARY. **Through any two points on a sphere (not the ends of a diameter) one and only one great circle may be drawn.**

§ 297. The two congruent parts into which a sphere is divided by a great circle are called ***hemispheres.*** Thus the equator is said to divide the earth into the Northern Hemisphere and the Southern Hemisphere.

§ 298. On a sphere the ***distance between two points*** is measured on the minor arc of a great circle connecting the two points, since, as can be proved, this is the shortest distance on the sphere between the points.

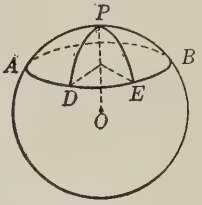
(1) Why do ships, going from New York to a European port due east, travel first a little north of east, then finally a little south of east?

EXERCISES

1. Tell whether the following are great circles or small circles of the earth considered as a sphere: the meridians, parallels of latitude, the equator.

2. What great circle separates the northern and southern hemispheres?

3. P is the pole of the small circle ADB, as shown. PD and PE are arcs of great circles. Prove that these arcs are equal.



(Ex. 3)

4. Prove: In a sphere chords which are equally distant from the center are equal.

5. Prove: In a sphere equal chords are equally distant from the center.

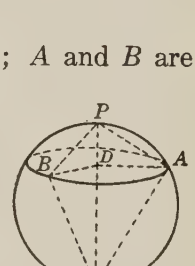
6. What are the circumference and the area of a circle of a sphere whose radius is 15 inches if the circle is 12 inches from the center of the sphere?

7. A small circle of a sphere has a radius of 5 inches and is 12 inches from the center of the sphere. How far from the center of the sphere is a small circle whose radius is 12 inches?

8. Prove: If any point on a sphere is connected with the extremities of a diameter not through the point, a right triangle is formed.

9. PP' is the axis of the small circle D; A and B are points on this circle, as shown. Prove that $\triangle PAP' \cong \triangle PBP'$. If the diameter of the sphere is 13 inches and PD is 4 inches, how long are AD, AP, and AP'?

10. P and P' are poles of the small circle D; A and B are points on the circle; PAP' and PBP' are arcs of great circles. Prove that arc AP = arc BP.



(Exs. 9–10)

11. The vertex of a right circular cone, the radius of whose base is 8 inches, is at the center of a sphere whose radius is 10 inches. If the altitude of the cone is 6 inches, will all the points of the cone lie inside the sphere? (What is the test as to whether a point lies inside a sphere or not?)

12. In a sphere whose radius is 10 inches, the vertex of a right circular cone is at the center. If the radius of the base of the cone is 8 inches and its altitude is 8 inches, what is the area of the part of the base of the cone which is outside the sphere?

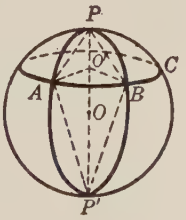
13. In the cone of Exercise 12, what is the total area outside the sphere?

14. If A and B are two fixed points on a sphere, what is the locus of the centers of all small circles which can be drawn through A and B? Prove that your statement is correct.

SUGGESTION. First consider the small circle through A and B perpendicular to the great circle determined by A and B. Revolve the plane of this circle about AB as an axis.

THEOREM 45

§ 299. The spherical distances of all points on a circle of a sphere from either pole of the circle are equal.



Given a circle ABC of a sphere, with P and P' its poles. Through A and B, any two points on the circle, arcs of great circles PAP' and PBP' are drawn.

To prove that arc AP = arc BP and arc AP' = arc BP'.

Proof. The proof is left for you.

Polar Distance

§ 300. The spherical distance from the nearer pole of a circle to any point on the circle is called the ***polar distance*** of the circle.

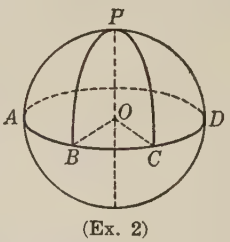
§ 301. One fourth of a great circle is called a ***quadrant.***

§ 302. COROLLARY. **The polar distance of a great circle is a quadrant.**

§ 303. COROLLARY. **The polar distances of equal circles on the same sphere or on equal spheres are equal.**

EXERCISES

1. If the extremities of a quadrant are connected with the center of the sphere, what kind of angle is formed?

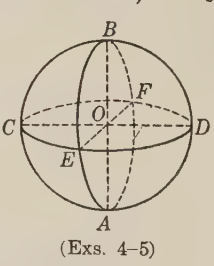


(Ex. 2)

2. Prove: If ABD is an arc of a great circle, PB and PC are quadrants, and O is the center of the sphere, then PO is perpendicular to the plane of the great circle ABD.

3. Represent on a sphere a small circle whose polar distance is one sixth of a great circle of the sphere. If the radius of the sphere is 10 inches, what is the radius of the small circle?

4. The figure at the right represents three great circles whose planes are perpendicular to each other and intersect in the lines AB, CD, and EF. Prove that BD is a quadrant.

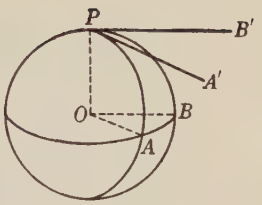


(Exs. 4–5)

5. If in this figure BEA, CED, and BCA are great circles and if BD and BE are quadrants, prove that BOA is an axis of circle CED.

6. On a material sphere whose diameter is 6 inches, a closed line is drawn by means of compasses set at a radius of 2 inches and revolved about any point of the sphere. Prove that the line is a circle. What is its circumference?

7. In the figure, P is the pole of the great circle AB; AP and BP are arcs of great circles to which $A'P$ and $B'P$ are tangent at the point P; O is the center of the sphere. If $\angle B'PA'$ is 35°, how many degrees are there in $\angle BOA$? How many degrees are there in arc AB? Does this relationship hold in general?

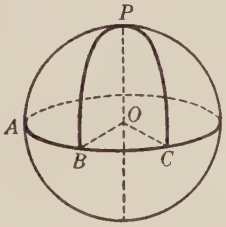


(Exs. 7–8)

8. In Exercise 7, how many degrees has the dihedral angle A–OP–B'?

THEOREM 46

§ 304. If two points on a sphere, not the extremities of a diameter, are each a quadrant's distance from a third point, the latter is a pole of the great circle passing through the former.



Given points B and C, not the extremities of a diameter, each a quadrant's distance from P on the sphere whose center is O. ABC is a great circle of the sphere drawn through B and C.

To prove that P is a pole of the great circle ABC.

Plan. Draw the diameter through P and the radii OB and OC. Show that PO is perpendicular to OB and OC and hence to the plane ABC.

Proof. The proof is left for you.

EXERCISE

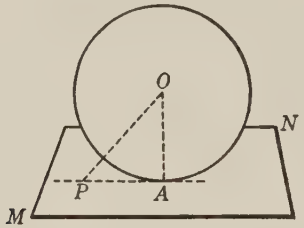
1. What is the locus of points on a sphere and at a quadrant's distance from a given point on the sphere?

Tangents to a Sphere

§ **305.** A straight line or a plane which has only one point in common with a sphere is said to be ***tangent to the sphere.*** The sphere is said to be tangent to such a line or plane. The point which is common to the sphere and the line or plane is called the ***point of tangency*** or the point of contact. Two spheres are said to be ***tangent to each other*** if they are tangent to the same plane at the same point.

Theorem 47

§ **306. A plane which is perpendicular to a radius of a sphere at its outer extremity is tangent to the sphere.**



Given plane MN perpendicular at A to the radius OA of sphere O.

To prove that plane MN is tangent to the sphere O.

Plan. Show that P, any point of MN other than A, is outside the sphere. How does the length of OP compare with the length of OA?

Proof. The proof is left for you.

§ **307.** Corollary. **A plane tangent to a sphere at a given point is perpendicular to the radius drawn to that point.**

Show that the radius is the shortest line from the center of the sphere to the plane. Since there can be only one shortest line from the center of the sphere to the plane, the plane must be perpendicular to the radius (§ 38).

(1) What relation does this corollary bear to Theorem 47?

EXERCISES

1. Prove: Two planes tangent to a sphere at the ends of a diameter are parallel.

2. Prove: At a point on a sphere there can be but one tangent plane.

3. Prove: If a line is perpendicular to a radius of a sphere at its outer extremity, it is tangent to the sphere.

4. If a line is tangent to a sphere, is it tangent to all the circles of the sphere through that point?

SUGGESTION. A line is not tangent to a circle if it is not in the plane of the circle.

5. Is a line which is tangent to a small circle of a sphere perpendicular to the radius of the sphere drawn to the point of contact?

6. Prove: A line which is tangent to a sphere is perpendicular to the radius drawn to the point of contact.

7. Prove: If a line is tangent to a sphere, it is tangent to all the circles made by planes passed through the line.

8. Prove: If two lines are tangent to a sphere at the same point, the plane determined by them is tangent to the sphere at that point.

9. Show how you would construct a plane through a given point on a sphere and tangent to it.

10. Prove: If a plane is tangent to a sphere, any line drawn in the plane through the point of contact is tangent to the sphere at that point.

11. Prove: If a line is tangent to a great circle of a sphere, it lies in the plane which is tangent to the sphere at that point.

12. Prove: Tangent lines drawn to a sphere from an external point are equal.

13. What is the locus of the points of contact of all tangent lines drawn to a sphere from an external point?

Review Exercises

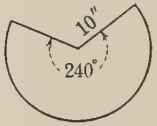
1. If the three face angles at one vertex of a parallelepiped are each 60° and the three edges meeting at that vertex are 2 inches, 3 inches, and 4 inches respectively, what is the total area of the parallelepiped?

2. Find the volume of the solid generated when a right triangle whose arms are 5 inches and 12 inches revolves about the hypotenuse.

3. How many cartloads will be required to remove a conical pile of dirt 12 feet high and 120 feet around the base? (Assume a cartload to be $1\frac{1}{2}$ cubic yards.)

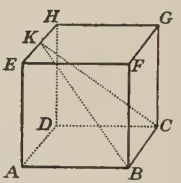
4. Prove: If a section of a frustum of a cone is made by a plane containing an element and one other point of the conical surface, the section is a trapezoid.

5. A cone is made from a sector of a circle whose central angle is 240°. If the radius of the sector is 10 inches, find the altitude, slant height, and radius of the base of the cone.



(Ex. 5)

6. AG is a cube whose edge is 16 inches. K is the midpoint of EH. Find the area of $\triangle BKC$.



(Ex. 6)

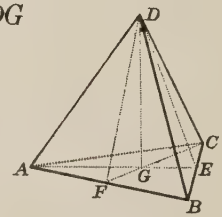
7. DG is the altitude of the regular tetrahedron D–ABC. AED and CFD are planes determined by DG and AD, and DG and CD, respectively.

(*a*) Prove that the plane determined by DG and AD is perpendicular to the plane DBC.

(*b*) Prove that a line from G perpendicular to plane DBC lies in plane AED.

(*c*) Prove that plane AED bisects the dihedral angle C–AD–B.

SUGGESTION. Through a point on DA construct a plane perpendicular to DA.



(Ex. 7)

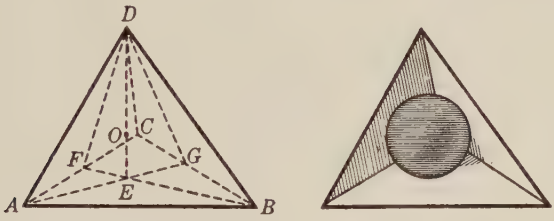
(*d*) Prove that all points on DG are equidistant from ABD, BCD, and ACD.

Inscribed and Circumscribed Spheres

§ 308. If a sphere is tangent to all the faces of a polyhedron, it is said to be ***inscribed*** in the polyhedron. The polyhedron is then said to be ***circumscribed*** about the sphere.

Theorem 48

§ 309. A sphere can be inscribed in any given tetrahedron.



Given any tetrahedron $ABCD$.

To prove that a sphere can be inscribed in $ABCD$.

Plan. Show that the planes bisecting the dihedral angles of the tetrahedron meet at a point which is equally distant from the four faces.

Proof.

1. Pass planes AGD and FBD bisecting dihedral angles DA and DB, respectively.
2. These planes will intersect in a line DE. (Why?)
3. Every point of DE, the intersection of these two planes, is equally distant from the faces ABD, BCD, and ACD. (Why?)
4. Pass a plane bisecting the dihedral angle AB.
5. This plane will meet DE at some point O. (Why?)
6. Point O is equally distant from all four faces. (Why?)
7. A sphere whose center is O and whose radius is equal to the distance from O to any one of the faces will be tangent to the four faces of the tetrahedron — i.e., inscribed in the tetrahedron $ABCD$.

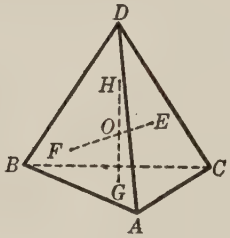
(1) Does the center of a sphere inscribed in a tetrahedron lie in the altitude of the tetrahedron?

(2) Show that only one sphere can be inscribed in any given tetrahedron.

§ 310. If all the vertices of a polyhedron lie on a spherical surface, the sphere is said to be ***circumscribed*** about the polyhedron. The polyhedron is then ***inscribed*** in the sphere.

THEOREM 49

§ 311. A sphere can be circumscribed about any given tetrahedron.



Given any tetrahedron *ABCD*.

To prove that a sphere can be circumscribed about *ABCD*.

Plan. Show that *EF* and *GH* (see Step 1) lie in the same plane and are not parallel. Hence their intersection *O* is equally distant from *A*, *B*, *C*, and *D*.

Proof.

1. At *E* and *G*, the centers of the circles about the faces *ADC* and *ABC*, construct *EF* and *GH* perpendicular to the faces.
2. Any point on *EF* is equally distant from *A*, *D*, and *C*. (Why?)
3. Any point on *GH* is equally distant from *A*, *B*, and *C*.
4. Any point on either *EF* or *GH* is equally distant from *A* and *C*.
5. *EF* and *GH* lie in the same plane (§ 103).
6. *EF* and *GH* are not parallel. (Suppose they were.)
7. Since *EF* and *GH* are in the same plane and are not parallel, they must meet at some point *O*.
8. *O* is equally distant from *A*, *B*, *C*, and *D* and is therefore the center of the required sphere.

§ 312. COROLLARY. **Through four points not in the same plane one spherical surface and only one can be passed.**

The Earth as a Sphere

The earth is not exactly a sphere. It is flattened out slightly at the poles. According to the most accurate measurements, the diameter of the earth from pole to pole is 7899 miles and the diameter at the equator 7925 miles.

For practical purposes, however, the earth may be regarded as a sphere. In order to locate positions on the earth accurately, two sets of circles are imagined to be drawn, as shown in Figure 1. A group of great circles are drawn through the poles. These are called ***meridians.*** The equator is a great circle whose plane is perpendicular to the planes of the meridians. Small circles are drawn with their planes parallel to the plane of the equator. These are called the ***parallels of latitude.*** The zero circles are the equator and the meridian through Greenwich, England.

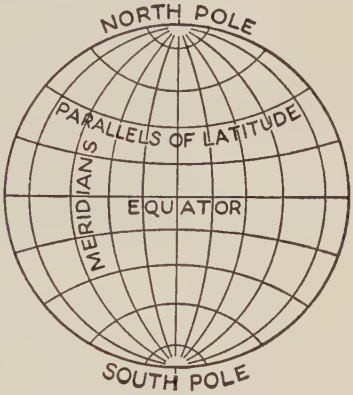


FIG. 1

A locality may be determined by the intersection of a meridian with a parallel of latitude. For example, if a place C (Figure 2) lies on a parallel 40° North and a meridian 70° West, it means that arc AC is 40° and arc AB is 70°. If O is the center of the earth, it is obvious that angle COA will measure the latitude of C and angle BOA its longitude.

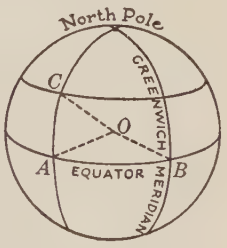


FIG. 2

At noon in any given place the sun is directly above the meridian of longitude of that place. This fact makes it quite simple to find the longitude of a given locality provided that one knows the difference in time between the locality and Greenwich. When it is noon at that locality, you know

how many hours before or after noon (depending on whether you are east or west of Greenwich) it is at Greenwich. Since the earth makes a complete rotation, 360°, in 24 hours, you can tell what part of a complete rotation the earth must make (or must have made) before (or after) noon at Greenwich, at which time the sun is directly above the Greenwich meridian.

For example, if it is 10 A.M. at Greenwich when it is noon at point A, the earth must rotate throughout $\frac{2}{24}$ of 360° before the sun will be directly above the Greenwich or zero meridian. Thus, point A is 30° east of Greenwich.

The latitude of a place in the Northern Hemisphere can be found by measuring the altitude of the Pole Star. The Pole Star is on a line with the north and south poles. Since light from the Pole Star seems to reach the earth in parallel rays, the observer at A will see the Pole Star in the direction indicated. (Figure 3.) OAD is perpendicular to EAB. Angle PAB is equal to $\angle AOT$, which is the latitude of A.

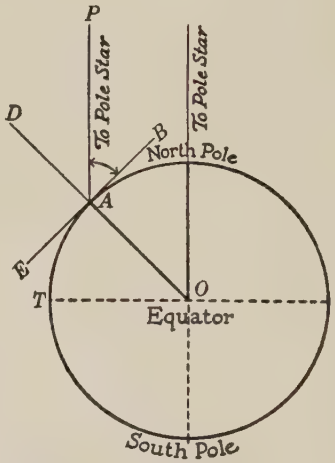


FIG. 3

You may also find your latitude by means of a home-made instrument. All that is needed is a metal tube for sighting placed along the base of a large protractor. A plumb line should be attached at O. (Figure 4.) DE is horizontal when the plumb line is over the 90° mark. When DE is sighted on the Pole Star and takes the new position $D'E'$, the angle DOD' is the altitude of the Pole Star and therefore your latitude. Angle AOB can be read from your protractor and is equal to angle DOD'.

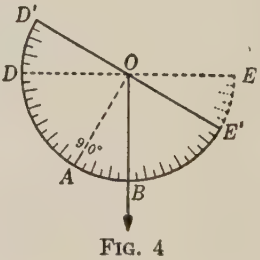


FIG. 4

Chapter Test

Complete the following sentences:

1. The lines OA and OB are radii of the same sphere. If OA is 3 inches, OB is ___?___ inches.

2. The radius of a sphere O is 4 inches; A is 3 inches from O. A is therefore ___?___ the sphere.

3. ABC is a section of a sphere O made by a plane. OD is perpendicular to the plane at D. If DC is 2 inches, DB is ___?___ inches.

4. If a plane is 12 inches from the center of a sphere whose radius is 13 inches, the radius of the section made by the plane is ___?___ inches.

5. A is the center of a small circle on sphere O. OB is perpendicular to the plane of the circle at B. Then A and B are the ___?___ point.

6. Two sections of a sphere are made by planes each 3 inches from the center of a sphere. If the radius of one section is 4 inches, the radius of the other section is ___?___ inches.

7. If A and B are the extremities of a diameter of a sphere, a section made by a plane passing through A and B is a ___?___ ___?___.

8. If A and B are the extremities of a diameter perpendicular to the plane of a small circle, A and B are the ___?___ of the circle.

9. In a sphere whose radius is 8 inches, the diameter of a great circle is ___?___ inches.

10. P is the pole of small circle ADB. If the great circle arc PD is 25°, the great circle arc PB is ___?___°.

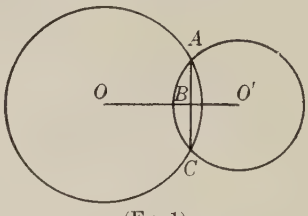
11. That part of the intersection of the planes of two great circles which is included within a sphere whose radius is 5 inches is ___?___ inches.

12. P and P' are the extremities of a diameter of a sphere. A is any point on the sphere. If the plane angle APP' is 25°, plane angle $PP'A$ is ___?___°.

13. If sphere O is tangent to plane MN at A and BAC is a line in MN, angle OAC is ___?___°.

Miscellaneous Exercises

1. Prove: The intersection of two spherical surfaces is a circle whose plane is perpendicular to the line joining the centers of the spheres and whose center is on that line.



(Ex. 1)

SUGGESTION. Pass a plane through the centers of the spheres and any point A on the intersection of the spheres. Then rotate this plane about OO' as an axis. What curve does the point A generate? Will this line be the intersection of the two spheres? What surface does the line AC generate?

2. Two spheres whose centers are 10 inches apart have radii 6 inches and 8 inches respectively. Find the circumference of their intersection.

3. Two spheres whose centers are 12 inches apart have radii 10 inches and 6 inches respectively. Find the area of their intersection.

4. Prove: Equal circles on the same sphere have equal polar distances.

5. If the radius of a sphere is 10 inches and the area of a section made by a plane is 201 square inches, what is the distance of the plane from the center of the sphere?

6. AB is a diameter of a sphere whose radius is 8 inches. At what point of AB should a plane be passed perpendicular to AB in order to get a section whose radius is 4 inches?

7. In a sphere whose diameter is 20 inches, a right circular cone is inscribed in such a way that the vertex of the cone is on the surface of the sphere and the base is a circle of the sphere. If the altitude of the cone is 16 inches, what is its lateral area?

8. What is the locus of the centers of spheres of a given radius which are tangent to a given plane?

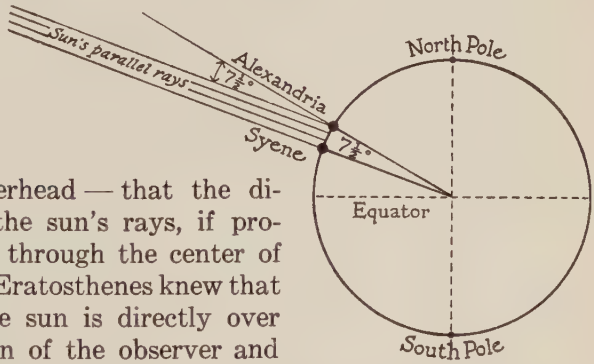
9. What is the locus of the centers of spheres which are tangent to a plane at a given point?

CHAPTER IX

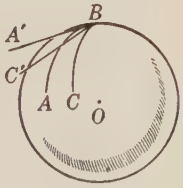
SPHERICAL POLYGONS — SPHERICAL VOLUMES

Early Measurement of the Earth

As early as about 200 B.C., Eratosthenes, the famous Greek scientist and mathematician, measured the circumference of the earth. Eratosthenes came upon some records which informed him that at noon at Syene, near the present Assuan Dam on the Nile River, on a certain day the sun was reflected from a deep well. This showed that the sun was



directly overhead — that the direction of the sun's rays, if produced, was through the center of the earth. Eratosthenes knew that at noon the sun is directly over the meridian of the observer and that Alexandria is on the same meridian and 500 miles north of Syene. At noon on the same day that the sun was reflected in the deep well at Syene, Eratosthenes found that the sun was $7\frac{1}{2}°$ south of the vertical at Alexandria. A study of the figure will show that this means that a $7\frac{1}{2}°$ arc is 500 miles long. The circumference of the earth can be easily calculated from this information.



§ **313.** When the arcs of two great circles meet on a sphere, the figure formed is called a ***spherical angle***. The measure of the angle is obtained by drawing the tangents ($A'B$ and $C'B$ in the figure) to the great circles at the vertex of the angle and measuring the angle thus formed.

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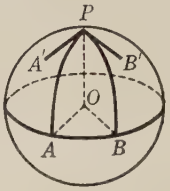
2-1
GEORGE EDWARD MAYCOCK

THEOREM 50

§ 314. An angle formed by arcs of two great circles has the same measure as the arc of a great circle described from the vertex of the angle as a pole and included between its sides, extended if necessary.



Given the spherical angle APB formed by the arcs AP and BP of two great circles. PA' and PB' are tangent to the arcs AP and BP respectively. AB is the arc of a great circle whose pole is P.

To prove that spherical $\angle APB$ = arc AB.[1]

Plan. Show that $\angle A'PB' = \angle AOB$.

Proof. Draw lines from P, A, and B to the center O.

STATEMENTS	REASONS
1. $A'P \perp OP$, $B'P \perp OP$.	1. Why?
2. Arcs AP and BP are quadrants.	2. § 302.
3. $\angle POA$ and $\angle POB$ are right angles.	3. Why?
4. $AO \perp OP$, $BO \perp OP$.	4. Why?
5. AO and $A'P$ are in the same plane. BO and $B'P$ are in the same plane.	5. Why?
6. $AO \parallel A'P$, $BO \parallel B'P$.	6. Why?
7. $\angle AOB = \angle A'PB'$.	7. § 60.
8. $\angle APB = \angle A'PB'$.	8. § 313.
9. $\angle AOB = \angle APB$.	9. Why?
10. $\angle AOB$ = arc AB.	10. Why?
11. $\angle APB$ = arc AB.	11. Why?

[1] Note that the sign of equality denotes the equality of the numerical measures, not the congruence of the figures.

§ 315. COROLLARY. **A spherical angle has the same measure as the dihedral angle formed by the planes of the two great circles.**

§ 316. COROLLARY. **All arcs of great circles drawn through the pole of a given great circle make right angles with the given circle.**

EXERCISES

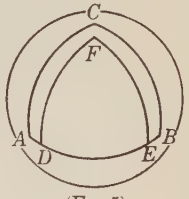
Use the figure for Theorem 50 for Exercises 1–4.

1. If arc AB is 60°, how many degrees are there in $\angle APB$? How many degrees are there in dihedral angle A–PO–B?

2. If arc AB is $a°$, how many degrees are there in $\angle APB$? How many degrees are there in dihedral angle A–PO–B?

3. If arc AB is 90°, how many degrees are there in $\angle APB$? How many degrees are there in dihedral angle A–PO–B?

4. How many degrees are there in arcs PB and PA and in dihedral angles P–OA–B and P–BO–A?

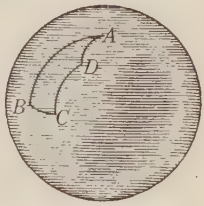


(Ex. 5)

5. In the figure at the right, AB, BC, CA, FD, and FE are arcs of great circles. A, B, and F are the poles of FE, FD, and AB respectively. If $\angle F = 50°$, how many degrees are there in arc DE? How many degrees are there in arc AE and in arc DB? How many degrees are there in arc AB? What is the sum of the degrees in $\angle F$ and arc AB?

Spherical Polygons

§ 317. A ***spherical polygon*** is a portion of a spherical surface bounded by three or more arcs of great circles. Thus, $ABCD$ is a spherical polygon. The arcs AB, BC, CD, and DA are the sides of the polygon. The points A, B, C, and D are the vertices of the polygon. The angles at A, B, C, and D are the angles of the polygon.



§ **318.** A spherical polygon is said to be ***convex*** if the corresponding polyhedral angle formed at the center of the sphere is convex. Only convex spherical polygons are treated in this book.

§ **319.** Such words as *diagonal, bisector, median, altitude,* etc., refer to arcs of great circles which have the same relations to spherical polygons that straight lines similarly named have to plane polygons.

§ **320.** A ***spherical triangle*** is a spherical polygon of three sides. The words *right, obtuse, acute, equiangular, isosceles,* and *equilateral,* when they refer to spherical triangles, have the same meanings as when they refer to plane triangles.

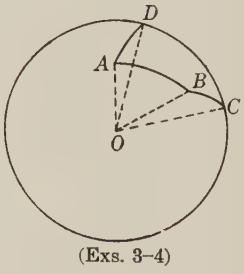
EXERCISES

1. In what sciences is the knowledge of spherical geometry of practical value?

2. Prove: The planes of the sides of a spherical polygon meet at the center of the sphere, forming a polyhedral angle.

3. $ABCD$ is a spherical polygon on a sphere whose center is O. If $\angle AOB$ is 48°, how many degrees are there in arc AB?

4. If dihedral angle AO in the figure is 60°, how many degrees are there in spherical angle DAB?



(Exs. 3–4)

5. Prove: The arcs which form the sides of a spherical polygon have the same measure as the corresponding face angles of the polyhedral angle at the center of the sphere.

6. Is it possible to have a spherical triangle whose sides are 90°, 52°, 67°?

7. Is it possible to have a spherical triangle whose sides are 120°, 150°, 200°?

8. Prove: The angles of a spherical polygon have the same measure as the corresponding dihedral angles of the polyhedral angle formed at the center of the sphere.

9. What is the relation between the sum of any two face angles of a trihedral angle and the third face angle? What is the corresponding proposition concerning the sides of a spherical triangle?

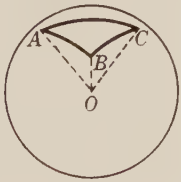
10. What is the proposition concerning the sum of the face angles of a convex polyhedral angle? What is the corresponding proposition concerning spherical polygons?

11. How many dihedral angles of a trihedral angle may be right angles? What is the corresponding fact concerning spherical triangles?

12. What proposition concerning spherical triangles corresponds to § 95?

THEOREM 51

§ 321. Any side of a spherical triangle is less than the sum of the other two sides.



Given the spherical triangle ABC, on a sphere whose center is O.

To prove that $AC < AB + BC$.

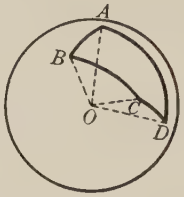
Plan. Draw the radii AO, BO, and CO, forming a trihedral angle at O. Since $\angle AOC < \angle AOB + \angle BOC$ (§ 93), $AC < AB + BC$ (Syllabus 99).

Proof. The proof is left for you.

§ 322. From any property of polyhedral angles we may infer an analogous property of spherical polygons, and conversely.

THEOREM 52

§ 323. The sum of the sides of a convex spherical polygon is less than 360°.



Given the convex spherical polygon $ABCD$, on sphere O.

To prove that the sum of the sides of $ABCD < 360°$.

Plan. Use § 94 and Syllabus 99.

Proof. The proof is left for you.

EXERCISES

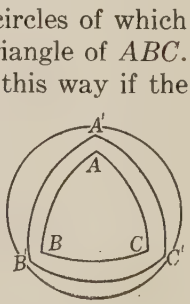
1. If two sides of a spherical triangle are 87° and 43° respectively, how many degrees may the third side contain?

2. May a spherical quadrilateral have quadrants for four of its sides?

Polar Triangles

§ 324. If from the vertices of a spherical triangle as poles arcs of great circles are drawn, another spherical triangle is formed which is called the ***polar triangle*** of the first.

If $A'B'$, $B'C'$, and $C'A'$ are arcs of great circles of which C, A, and B are poles, $A'B'C'$ is the polar triangle of ABC. More than one triangle would be formed in this way if the arcs $A'B'$, $B'C'$, and $C'A'$ were completed. How many? Of these triangles the only one which is called the polar triangle of ABC is the one for which A and A' lie on the same side of BC, B and B' on the same side of AC, and C and C' on the same side of AB.



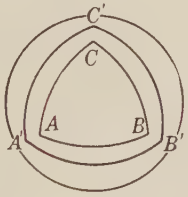
Preparation for Theorem 53. (1) If, in the figure below, $A'B'C'$ is the polar triangle of ABC, show that A is a quadrant's distance from any point of $B'C'$.

(2) If, in the figure below, $A'B'C'$ is the polar triangle of ABC, show that B is a quadrant's distance from any point of $A'C'$.

(3) In Exercises 1 and 2 you have shown that C' is a quadrant's distance from both A and B. What relation has C' to AB? Show that A' is a quadrant's distance from B and C, and B' a quadrant's distance from A and C.

Theorem 53

§ 325. If one spherical triangle is the polar triangle of another, then, reciprocally, the second is the polar triangle of the first.



Given $\triangle A'B'C'$, the polar triangle of $\triangle ABC$.

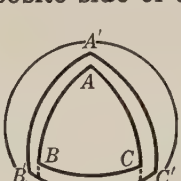
To prove that $\triangle ABC$ is the polar triangle of $\triangle A'B'C'$.

Proof.

Statements	Reasons
1. A is the pole of $B'C'$ and B is the pole of $A'C'$.	1. § 324.
2. The distances from A to C' and from B to C' are each quadrants.	2. § 302.
3. Since C' is a quadrant's distance from A and B, C' is the pole of AB. Likewise, B' and A' may be shown to be the poles of AC and BC respectively.	3. § 304.
4. $\triangle ABC$ is the polar triangle of $A'B'C'$.	4. § 324.

Theorem 54

§ 326. In two polar triangles any angle of either is the supplement of the opposite side of the other.



Given $\triangle A'B'C'$, the polar triangle of $\triangle ABC$.

To prove that $\angle A + B'C' = 180°$, $\angle B' + CA = 180°$,
$\angle A' + BC = 180°$, $\angle C + A'B' = 180°$,
$\angle B + C'A' = 180°$, $\angle C' + AB = 180°$.

Proof. Extend the arcs AB and AC until they meet $B'C'$ at D and E respectively.

Statements	Reasons
1. $A'B'C'$ is the polar triangle of $\triangle ABC$.	1. Given.
2. ABC is the polar triangle of $\triangle A'B'C'$.	2. Why?
3. B' is the pole of AE, and $B'E = 90°$. Similarly, $DC' = 90°$.	3. §§ 324, 302.
4. $B'E + DC' = 180°$.	4. Why?
5. $B'E = DE + B'D$.	5. Why?
6. $DE + B'D + DC' = 180°$.	6. Why?
7. A is the pole of DE.	7. § 324.
8. $DE = \angle A$.	8. § 314.
9. $\angle A + B'C' = 180°$.	9. Why?

Similarly, the other relations may be proved.

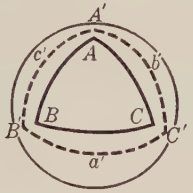
EXERCISES

1. Find the number of degrees in the sides of a spherical triangle when the angles of its polar triangle are as follows:

(*a*) 87°, 72°, 58°
(*b*) $72\frac{1}{2}°$, $96\frac{3}{4}°$, $56\frac{1}{2}°$
(*c*) 83° 57′, 82° 29′, 68° 7′
(*d*) 43.2°, 89.4°, 103.6°

THEOREM 55

§ 327. The sum of the angles of a spherical triangle is greater than 180° and less than 540°.



Given the spherical triangle ABC.

To prove that $\angle A + \angle B + \angle C > 180°$ and $< 540°$.

Proof. Draw $A'B'C'$, the polar triangle of ABC, and denote the number of degrees in its sides by a', b', c'.

STATEMENTS	REASONS
1. $\angle A + a' = 180°$, $\angle B + b' = 180°$, $\angle C + c' = 180°$.	1. § 326.
2. $\angle A + \angle B + \angle C + a' + b' + c' = 540°$.	2. Why?
3. $a' + b' + c' < 360°$.	3. § 323.
4. $\angle A + \angle B + \angle C > 180°$.	4. Why?
5. $\angle A + \angle B + \angle C + a' + b' + c' = 540°$.	5. Step 2.
6. $a' + b' + c' > 0°$.	6. Why?
7. $\angle A + \angle B + \angle C < 540°$.	7. Why?

EXERCISES

1. Prove: The polar triangle of an isosceles triangle is isosceles.

2. Prove: If a spherical triangle is equiangular, its polar triangle is equilateral; and, conversely, if a spherical triangle is equilateral, its polar triangle is equiangular.

3. Prove: If three sides of one spherical triangle are equal to three sides of another, the three angles of the polar triangle of one are equal to the three angles of the polar triangle of the other; and conversely.

Angles of a Spherical Triangle

§ **328.** Since the sum of the angles of a spherical triangle may be any number of degrees between 180° and 540°, a spherical triangle may contain one, two, or three right angles. It may contain also one, two, or three obtuse angles. Compare these facts with the corresponding facts in plane geometry. A spherical triangle containing two right angles is called a ***birectangular*** triangle. One containing three right angles is called a ***trirectangular*** triangle. Make a drawing of each of these two kinds of triangles.

EXERCISES

1. Prove: The sides opposite the right angles in a birectangular spherical triangle are quadrants and the third side measures the opposite angle.

2. Prove: The polar triangle of a birectangular spherical triangle is birectangular.

3. Prove: Each side of a trirectangular spherical triangle is a quadrant.

4. Represent a trirectangular spherical triangle and its polar triangle.

5. Is each angle of an equilateral spherical triangle 60°?

6. If you know the size of two angles of a spherical triangle, can you find the size of the third angle?

Review Exercises

1. If from a point 8 inches away from the center of a sphere whose radius is 4 inches all possible tangents to the sphere are drawn, how long is the locus of the points of contact?

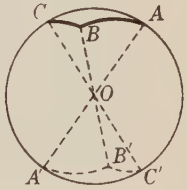
2. Prove: If a plane is tangent to a sphere, a line perpendicular to the plane from the center of the sphere passes through the point of contact.

3. Prove: If a plane is tangent to a sphere, a line perpendicular to the plane at the point of contact passes through the center of the sphere.

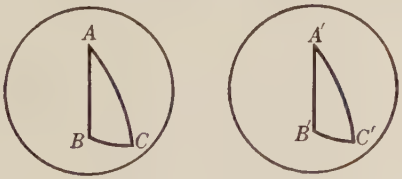
Congruent Spherical Triangles

§ 329. ***Congruent spherical triangles*** are triangles whose sides and angles are equal each to each and arranged in the same order; that is, spherical triangles which can be made to coincide.

§ 330. Symmetric figures were defined in § 96. Two spherical triangles are symmetric if their sides and angles are equal each to each but arranged in the opposite order. If diameters are drawn to each vertex of a spherical triangle ABC as shown and the points A', B', C' are connected by arcs of great circles, $A'B'C'$ is symmetric to ABC. Why? May two symmetric triangles be made to coincide?



§ 331. POSTULATE. **Two spherical triangles on the same sphere or on equal spheres are congruent if (*a*) two sides and the included angle of one are equal respectively to two sides and the included angle of the other, or (*b*) two angles and the included side of one are equal respectively to two angles and the included side of the other, and the parts are arranged in the same order.**



§ 332. COROLLARY. **Two spherical triangles on the same sphere or on equal spheres are symmetric if (*a*) two sides and the included angle of one are equal respectively to two sides and the included angle of the other, or (*b*) two angles and the included side of one are equal respectively to two angles and the included side of the other, and the parts are arranged in the opposite order.**

THEOREM 56

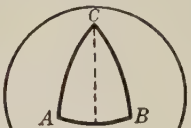
§ 333. Two triangles on the same sphere or on equal spheres are either congruent or symmetric if the three sides of one are equal respectively to the three sides of the other.

Draw a radius to each vertex. Use §§ 95 and 315.

§ 334. COROLLARY. **Two triangles on the same sphere or on equal spheres are either congruent or symmetric if the three angles of one are equal respectively to the three angles of the other.**

Draw the polar triangle of each. See Exercise 3, page 195. How does this proposition differ from the corresponding one in plane geometry?

§ 335. COROLLARY. **If two sides of a spherical triangle are equal, the angles opposite these sides are equal.**



Join the vertex C with the middle point of the base.

§ 336. COROLLARY. **If two angles of a spherical triangle are equal, the sides opposite these angles are equal.**

Draw the polar triangle.

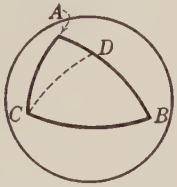
§ 337. COROLLARY. **If a spherical triangle is equilateral, it is also equiangular; and if it is equiangular, it is also equilateral.**

§ 338. COROLLARY. **Two symmetric isosceles spherical triangles are congruent.**

Since the triangles are symmetric, the parts are equal but arranged in the reverse order. But if, in spherical triangles ABC and $A'B'C'$, $AC = BC$ and $A'C' = B'C'$, $\angle A = \angle B$ and $\angle A' = \angle B'$. The parts may be considered as equal in the same order and are therefore congruent.

THEOREM 57

§ 339. If two angles of a spherical triangle are unequal, the sides opposite these angles are unequal in the same order.



Given the spherical triangle ABC, with $\angle C > \angle B$.

To prove that $AB > AC$.

Proof. Draw the arc CD of a great circle, making $\angle DCB = \angle B$.

STATEMENTS	REASONS
1. $DB = DC$.	1. Why?
2. $AD + DC > AC$.	2. § 321.
3. $AD + DB > AC$.	3. Why?
4. $AD + DB = AB$.	4. Why?
5. $AB > AC$.	5. Why?

§ 340. COROLLARY. **If two sides of a spherical triangle are unequal, the angles opposite these sides are unequal in the same order.**

SUGGESTION. Use the indirect method of proof.

EXERCISES

1. Prove: If a circle is drawn through the vertices of a spherical triangle and the spherical distances from each vertex to the pole of this circle are drawn, three isosceles spherical triangles are formed.

2. Prove: If circles are circumscribed about two congruent plane triangles, the circles are equal.

3. Prove: Any point on the bisector of a spherical angle is equally distant from the sides of the angle.

THEOREM 58

§ 341. Two symmetric spherical triangles are equal in area.

Given the two symmetric spherical triangles ABC and $A'B'C'$.

To prove that $\triangle ABC = \triangle A'B'C'$.[1]

Proof.

STATEMENTS	REASONS
1. Pass planes through points A, B, and C and through points A', B', and C', thus forming the small circles ABC and $A'B'C'$.	1. §§ 11, 278.
2. Since the arcs AB, BC, and CA are equal, respectively, to arcs $A'B'$, $B'C'$, and $C'A'$, the chords AB, BC, and CA are equal to the chords $A'B'$, $B'C'$, and $C'A'$.	2. Why?
3. Plane $\triangle ABC \cong$ plane $\triangle A'B'C'$.	3. Why?
4. Circle $ABC =$ circle $A'B'C'$.	4. Why?
5. If P and P' are the poles of circles ABC and $A'B'C'$, the arcs PA, PB, PC, $P'A'$, $P'B'$, and $P'C'$ are all equal to each other.	5. § 303.
6. $\triangle APB = \triangle A'P'B'$, $\triangle APC = \triangle A'P'C'$, $\triangle BPC = \triangle B'P'C'$.	6. §§ 333, 338.
7. $\triangle APB + \triangle APC + \triangle BPC = \triangle A'P'B' + \triangle A'P'C' + \triangle B'P'C'$. Or $\triangle ABC = \triangle A'B'C'$.	7. Why?

[1] Note that the sign of equality denotes equality of area, not congruence.

Comparison of Plane and Spherical Geometry

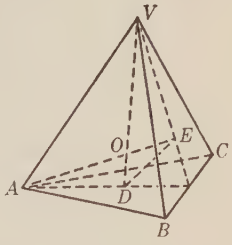
§ 342. In spherical geometry an arc of a great circle corresponds to a straight line in plane geometry. Thus in plane geometry the shortest line between two points is a straight line; in spherical geometry the shortest line between two points is the arc of a great circle.

A straight line is determined by any two of its points; an arc of a great circle is likewise determined by two points (except that the two points must not be at the extremities of a diameter). There is this important difference, however. Straight lines may be parallel, but arcs of great circles are never parallel. For this reason, theorems in spherical geometry and the corresponding theorems in plane geometry which depend upon parallel lines are different. For instance, the sum of the angles of a plane triangle is 180°. This theorem, as you know, is not true of spherical triangles.

Another difference between plane and spherical geometry is due to the curvature of the sphere. Plane symmetric triangles can be made to coincide, while spherical symmetric triangles cannot in general be made to coincide. If you are careful to avoid theorems which do not depend upon these two differences, you may find many theorems which are the same in both plane and spherical geometry.

Miscellaneous Exercises

1. If lines are drawn from each vertex of a tetrahedron to the intersection of the medians of the opposite face, they meet at a point that divides each line in the ratio 3 : 1. (Draw DE and prove it parallel to AV. See figure.)



(Ex. 1)

2. A conical container of slant height 12 inches is made to hold exactly 4 quarts of water. How far from the vertex should a mark be placed to indicate the contents of two quarts?

3. Prove: If an arc of one great circle passes through the pole of another, the two arcs meet at right angles.

4. Prove: On a given sphere the smallest circle whose plane passes through a given point within the sphere is the one whose plane is perpendicular to the radius drawn through the given point.

5. Prove: Any point on the arc of a great circle which bisects an arc of another great circle at right angles is equally distant from the extremities of the second arc.

6. Show how to draw an arc of a great circle through a point on a sphere so that it will make right angles with the arc of a given great circle.

7. What is the locus of the poles of great circles that pass through the extremities of a diameter of a sphere?

8. A regular pyramid with a square base 3 inches on a side is inscribed in a sphere whose radius is 12 inches. What is the volume of the pyramid?

9. Show how to construct a sphere of a given radius so that it will pass through three given points.

10. Show how to construct a sphere of a given radius so that it will pass through two given points and will be tangent to a given plane.

11. When an aviator reaches a point 2 miles above the earth, what is the radius of his horizon? (Consider the earth a sphere whose radius is 4000 miles.)

12. Prove: The line of centers of two tangent circles passes through the point of contact.

13. Two spheres, one 4 inches and the other 8 inches in diameter, rest on the same table in contact with each other. How far apart are the points of support?

14. Explain how you would bisect a spherical angle.

15. Find the number of degrees in the angles of a spherical triangle when the number of degrees in the sides of its polar triangle are 78°, 93°, and 65° respectively.

Spherical Areas

§ 343. If a regular polygon is inscribed in a circle and the number of sides is indefinitely increased, the polygon, as you know, soon becomes indistinguishable from the circle. Any desired approximation to the circumference and to the area of the circle may be obtained by such a process, and by the use of the theory of limits an exact formula for these measures is obtained. Similarly, any desired approximation to the area and to the volume of a sphere may be found.

Inscribe in a semicircle half of a regular polygon of an even number of sides, as shown in Figure 1. If the figure is revolved about AE as an axis, what is the nature of the surfaces generated by AB, BC, CD, and DE respectively? Now imagine the number of sides increased indefinitely by some process such as repeated doubling. Since the semipolygon will approach more and more closely to the semicircle, the surface generated by the semipolygon will approach more and more closely to the surface generated by the semicircle — that is, to the surface of a sphere. We have therefore a method of approximating the measure of the surface of a sphere.

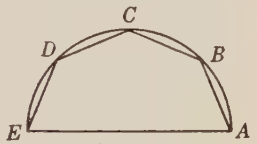


FIG. 1

$ABCD$ in Figure 2 is half of a regular hexagon inscribed in a semicircle. FO is the perpendicular bisector of AB. FG and BH are perpendicular to AD.

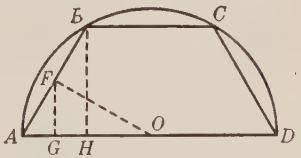


FIG. 2

(1) What kind of a surface is generated by AB when the figure is revolved about AD?

(2) In terms of BH and AB find the area generated by AB when the figure is revolved about AD as an axis.

(3) Prove: $AB \times FG = AH \times FO$, and that $AB \times BH = 2\,AH \times FO$.

§ 344. POSTULATE. **If half of a regular polygon of an even number of sides is inscribed in a semicircle and the number of sides is increased indefinitely, the surface generated by the semipolygon revolving about the diameter of the semicircle as an axis approaches as a limit the spherical surface generated by the semicircle.**

§ 345. The *area of a sphere* is the limit of the areas of the surfaces generated by the sides of a semipolygon inscribed in a semicircle, as they revolve about the diameter of the semicircle. With your present knowledge, the computation of the area of a spherical surface would be a laborious task. After you have studied the next proposition, you will find the computation much simpler. The exercises which follow will help to prepare you for this proposition.

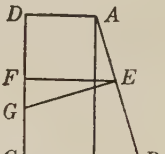


FIG. 1

(1) In the quadrilateral $ABCD$ (Figure 1) the lines AD, EF, and BC are perpendicular to DC, AH is perpendicular to BC, and EG is perpendicular to AB. Prove that $AB \times EF = AH \times EG$.

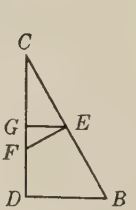


FIG. 2

(2) In the triangle BCD (Figure 2) the lines EG and BD are perpendicular to CD, and EF is the perpendicular bisector of BC. Prove that $BC \times EG = CD \times EF$, and that $BC \times DB = 2\,CD \times EF$.

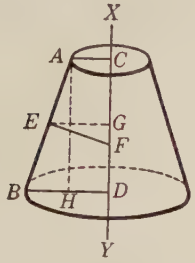
(3) Review Exercises (2) and (3) on page 203 and show that the area generated by AB as it revolves about AD may be found by either of the two formulas:

$$L = \pi BH \times AB$$
$$\text{or } L = AH \times 2\,\pi FO.$$

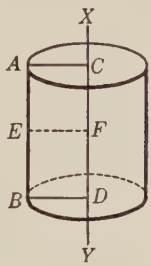
(4) In Figure 2 on page 203, AD is 16 inches. Find the area generated by AB in two ways, by using both the formulas in Exercise (3).

THEOREM 59

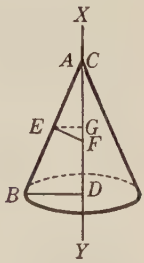
§ 346. The area of the surface generated by a line segment revolving about an axis in its plane, not perpendicular to it or crossing it, is equal to the product of the projection of the segment on the axis and the circumference of the circle whose radius is the perpendicular drawn at the middle point of the segment and terminated by the axis.



CASE I



CASE II



CASE III

Given line AB revolving about XY, which is in the same plane with it but is not perpendicular to it and does not cross it. CD is the projection of AB on XY. EF is the perpendicular bisector of AB. Let S denote the area generated by AB.

To prove that $S = CD \times 2\,\pi EF$.

Proof.

CASE I. *AB is not parallel to XY and does not meet XY.*

In this case S is the lateral area of a frustum of a cone of revolution. Draw $EG \perp XY$ and $AH \perp BD$.

STATEMENTS	REASONS
1. $S = AB \times 2\,\pi EG$.	1. § 266, Syllabus 43.
2. $\triangle ABH \sim \triangle EFG$.	2. Syllabus 131.
3. $AB : EF = AH : EG$.	3. Why?
4. $AB \times EG = AH \times EF$.	4. Why?
5. $AB \times EG = CD \times EF$.	5. Why?
6. $S = CD \times 2\,\pi EF$.	6. Why?

CASE II. *AB is parallel to XY.*

In this case S is the lateral area of a cylinder of revolution and is therefore equal to $CD \times 2\pi EF$.

CASE III. *AB is not parallel to XY and meets it at A.*

In this case S is the lateral area of a cone of revolution. (The proof is left for you.)

EXERCISES

1. Find the area of the surface generated by revolving AB (in the figure for Case I, page 205) about XY when $AB = 20$, $\angle HAB = 30°$, and $EF = 8$.

2. Find the area of the surface of the frustum of the cone of Exercise 1 by finding the radii of the two bases and substituting in the formula $L = \pi s(r + r')$. Is your answer the same as that for Exercise 1?

3. In trapezoid $ABCD$, $\angle A$ and $\angle B$ are right angles, $BC = 12$, $CD = 24$, and $DA = 8$. Compute the area generated by revolving CD about AB.

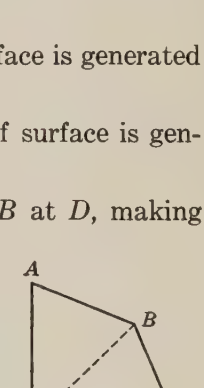
4. If AB intersects XY, what kind of surface is generated by revolving AB about XY?

5. If AB is parallel to XY, what kind of surface is generated by revolving AB about XY?

6. AB is 20 inches long. XY bisects AB at D, making $\angle BDX = 120°$. What is the area of the surface generated by revolving AB about XY?

7. In the isosceles trapezoid $ABCD$, $\angle A = 60°$ and $AB = 16$ inches. The equal sides of the trapezoid are each 12 inches. Compute the area of the surface generated by revolving the trapezoid about the base AB.

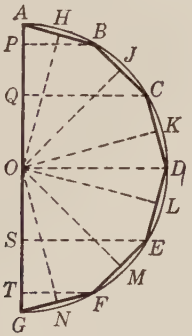
8. Figure $ABCDE$ is half of a regular octagon. What must be known in order to compute the area of the surface generated by revolving the figure about AE?



(Ex. 8)

THEOREM 60

§ 347. The area of the surface generated by half of a regular polygon of an even number of sides inscribed in a semicircle is equal to the product of the diameter of the semicircle and the circumference of a circle whose radius is the perpendicular from the center to any one of the sides.



Given $ABCD \cdots$, half of a regular polygon of an even number of sides inscribed in the semicircle ADG and revolving about AG as an axis. Let S denote the area of the surface generated by $ABCD \cdots$ and a the length of the perpendicular from O to AB.

To prove that $S = AG \times 2\pi a$.

Plan. By § 346 the area of the surface generated by AB is $AP \times 2\pi HO$; the area of the surface generated by BC is $PQ \times 2\pi JO$; the area of the surface generated by CD is $QO \times 2\pi KO$; the area of the surface generated by DE is $OS \times 2\pi LO$; etc. Since HO, JO, KO, LO etc., are equal, they may be denoted by a. The total surface is then $2\pi a(AP + PQ + QO + OS \cdots) = 2\pi a \times AG$.

Proof. The proof is left for you.

Go through the same reasoning with half of a regular polygon of 24 sides inscribed in the semicircle. You will get the same results.

§ **348.** Theorem 60 is true, no matter how many sides there may be in the semipolygon. If the number is increased indefinitely, the length of the perpendicular a approaches the radius r of the semicircle as a limit. The limit of the areas generated by the semipolygon is the area of the sphere generated by the semicircle. We are led therefore to the following statement:

§ **349.** POSTULATE. **The area of a sphere is equal to the product of the diameter and the circumference of a great circle.**

§ **350.** COROLLARY. **The area S of a sphere is equal to that of four great circles.**

$$S = 4\pi r^2$$

§ **351.** COROLLARY. **The areas of two spheres are to each other as the squares of their radii or diameters.**

EXERCISES

1. Find the areas of spheres whose radius is:

5 in.; 8.2 in.; $4\frac{1}{2}$ ft.; 2 ft. 3 in.; 8.44 ft.; 96.2 ft.

2. Solve $S = 4\pi r^2$ for r.

3. Find the radii of spheres whose areas are as follows:

(a) 2464 sq. in. (b) 12.57 sq. ft.

4. What is the ratio of the radii of two spheres if the area of one is twice the area of the other?

5. Find the length of the radius of a sphere whose area is one third that of a sphere whose radius is 24 inches.

6. If a hemispherical dome is 6 yards in diameter, how many square feet of tin are necessary to cover it?

7. Prove that the area of a sphere is equal to the lateral area of the circumscribed right circular cylinder. (The bases and the elements of the cylinder are tangent to the sphere.)

8. What is the area of the earth considered as a sphere whose radius is 4000 miles?

Zones

§ 352. A ***zone*** is a portion of a spherical surface included between two parallel planes. The circles made by the two planes are the ***bases of the zone.*** The distance between the planes is the ***altitude of the zone.*** If one of the planes is tangent to the sphere, the resulting zone is called a ***zone of one base.***

§ 353. If a semicircle such as PCQ in Figure 1 is revolved about the diameter PQ as an axis, the arc AC generates a zone whose bases are the circles AB and CD and whose altitude is EF. The arc PA generates a zone of one base whose altitude is PE.

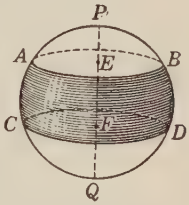


FIG. 1

A zone is a surface, not a solid.

§ 354. The area of a zone may be found by reasoning similar to that used for finding the area of a sphere. If the arc AB (Figure 2) is revolved about the diameter PQ, the surface generated by the chord AB is $CD \times 2\pi EO$, where EO is a line from the center perpendicular to the chord.

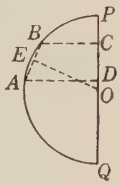


FIG. 2

If the number of chords from A to B is increased indefinitely, EO will approach the length of the radius as a limit and the sum of the chords will approach arc AB. No matter how many chords there are, the surface generated will be given by the formula $S = CD \times 2\pi EO$. We are led therefore to the following statement:

§ 355. POSTULATE. **The area S of a zone whose altitude is h is given by the formula**

$$S = 2\pi rh.$$

§ 356. COROLLARY. **The area of a zone is equal to the product of the altitude of the zone and the circumference of a great circle.**

EXERCISES

1. State which of the following are zones of one base: torrid zone, north temperate zone, south temperate zone, Arctic zone, Antarctic zone. What are the names of the bases of these zones?

2. If the altitude of a zone on a sphere is doubled, in what ratio is the area increased?

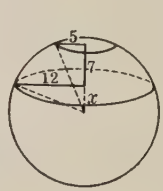
3. On a sphere whose radius is 20.3 inches, find the areas of zones whose altitudes are as follows: (*a*) 4.3 inches, (*b*) 7.23 inches, (*c*) 6.38 inches.

4. If the area of a zone of one base on a sphere whose radius is 6 inches is 24π, what is the diameter of the base?

5. Show that the area of a zone of one base made by revolving an arc AB is equal to the area of a circle whose radius is the chord AB.

6. If the area of a sphere is 144π square inches and the area of a zone on the sphere is 24π square inches, what is the radius of the sphere and the altitude of the zone?

7. The radii of the bases of a zone are 5 inches and 12 inches. Its altitude is 7 inches. Find its area.



(Ex. 7)

8. The radii of the bases of a zone are 6 inches and 8 inches. Its altitude is 2 inches. Find its area.

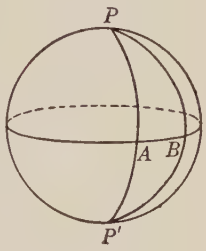
9. Find the area of the earth's surface between the parallels of latitude 30° North and 45° North. (Consider a great circle made by any plane perpendicular to the plane of the equator. What angle is subtended at the center by an arc of this great circle between the equator and the parallel of latitude 30° North?)

10. If the radius of the base of a zone of one base is 6 feet and the altitude of the zone is 3 feet, what is the diameter of the sphere?

11. If a point of light is placed 2 inches from a sphere whose radius is 3 inches, how many square inches of the sphere are illuminated by it?

Lunes

§ **357.** A ***lune*** is a portion of a sphere included between the halves of two great circles. In the figure, $PAP'B$ is a lune. The angle APB or $AP'B$ between the semicircles is called the ***angle of the lune.*** The notation "lune $\angle P$" will be used to denote a lune whose angle is P.



§ **358.** If two lunes on the same sphere have equal angles, they are congruent, for they can be made to coincide. From this it follows that a lune whose angle is 1° is $\frac{1}{360}$ part of the surface of the sphere, for 360 lunes whose angles are 1° will cover the whole sphere without overlapping. The whole sphere may be considered as a lune whose angle is 360°. A lune whose angle is 60° is $\frac{1}{6}$ of the spherical surface; a lune whose angle is 90° is $\frac{1}{4}$ of the spherical surface. We are led therefore to the following statement:

§ **359.** POSTULATE. **The area of a lune is to the area of the sphere as the angle of the lune is to 360°.**

EXERCISES

1. What part of the spherical surface is a lune whose angle is 10°? If the radius of the sphere is 4 inches, what is the area of the lune?

2. What part of the spherical surface is a lune whose angle is 54°? Find the area of such a lune on a sphere whose diameter is 18 inches.

3. On a sphere whose radius is 8 inches, find the areas of lunes whose angles are as follows:

(*a*) 60°	(*c*) 180°	(*e*) 22° 30′
(*b*) 45°	(*d*) 83°	(*f*) 46° 20′

4. What part of the earth's surface is included between the circles of longitude 20° West and 45° West? What is the approximate area of this surface?

5. The area of a lune is 50 square feet. If the area of the surface of the sphere is 250 square feet, what is the angle of the lune?

6. Show that the sum of the areas of a lune whose angle is 20° and a lune whose angle is 30° on a given sphere is equal to the area of a lune whose angle is 50°; that is, lune $\angle 20°$ + lune $\angle 30°$ = lune $\angle 50°$.

7. Show that half the area of a lune whose angle is 40° on a given sphere is equal to the area of a lune whose angle is 20°; that is, $\frac{1}{2}$ lune $\angle 40°$ = lune $\angle \frac{1}{2} \times 40°$.

Spherical Excess

§ 360. The ***spherical excess of a triangle*** is the difference between the sum of the angles of a spherical triangle and 180°. If A, B, and C represent the measures of the angles of a spherical triangle, the spherical excess is $A + B + C - 180°$.

Since the sum of the angles of a plane polygon of n sides is $(n - 2)180°$, the ***spherical excess of a spherical polygon*** is the difference between the sum of the angles of the polygon and $(n - 2)\ 180°$. If $A, B, C, D \cdots$ represent the measures of the angles of the polygon, the spherical excess is $(A + B + C + D + \cdots) - (n - 2)\ 180°$.

(1) Find the spherical excess of triangles whose angles are 100°, 125°, 140°; 72°, 95°, 67°; 115° 27′, 93° 32′, 86° 29′.

(2) What is the spherical excess of a triangle whose angles in degrees are A, B, and C?

(3) Show that the area of a hemisphere is the same as the area of a lune whose angle is 180°.

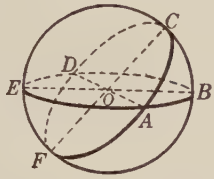
(4) Show that on a given sphere the sum of the areas of lunes whose angles in degrees are A, B, and C is equal to the area of a lune whose angle is $A + B + C$; that is, lune $\angle A$ + lune $\angle B$ + lune $\angle C$ = lune $\angle (A + B + C)$.

(5) Show that on a given sphere $\frac{1}{2}$ lune $\angle A$ = lune $\frac{1}{2} \angle A$.

(6) Show that on a given sphere $\frac{1}{2}$ (lune $\angle A$ + lune $\angle B$ + lune $\angle C$ − lune $\angle 180°$) = lune $\angle \frac{1}{2}(A + B + C - 180°)$.

THEOREM 61

§ 361. The area of a spherical triangle is equal to the area of a lune whose angle is half the spherical excess of the triangle.



Given the spherical triangle ABC on a sphere whose center is O.

To prove that $\triangle ABC = \text{lune } \angle \frac{1}{2}(A + B + C - 180°)$.

Plan. Note that the sum of the three lunes, lune $\angle A$, lune $\angle B$, and lune $\angle C$, includes the hemisphere A–$BCEF$ (when $\triangle AEF$ is substituted for $\triangle BCD$) and 2 $\triangle ABC$ in addition. Hence lune $\angle A$ + lune $\angle B$ + lune $\angle C$ = a hemisphere (or lune $\angle 180°$) + 2 $\triangle ABC$, and $\triangle ABC = \frac{1}{2}$ (lune $\angle A$ + lune $\angle B$ + lune $\angle C$ − lune $\angle 180°$) = lune $\angle \frac{1}{2}(A + B + C - 180°)$.

Proof. Extend the sides of $\triangle ABC$ to complete circles.

STATEMENTS	REASONS
1. Since $\triangle AEF$ and $\triangle BCD$ are symmetric, $\triangle AEF = \triangle BCD$.	1. §§ 330, 341.
2. Lune $\angle A = \triangle ABC + \triangle BCD$.	2. Why?
3. Lune $\angle A = \triangle ABC + \triangle AEF$.	3. Why?
4. Lune $\angle B = \triangle ABC + \triangle ACE$.	4. Why?
5. Lune $\angle C = \triangle ABC + \triangle ABF$.	5. Why?
6. Lune $\angle A$ + lune $\angle B$ + lune $\angle C$ = 2 $\triangle ABC + \triangle ABC + \triangle AEF + \triangle ACE + \triangle ABF$.	6. Why?
7. $\triangle ABC + \triangle AEF + \triangle ACE + \triangle ABF$ = hemisphere A–$BCEF$.	7. Why?
8. Lune $\angle A$ + lune $\angle B$ + lune $\angle C$ = 2 $\triangle ABC$ + hemisphere A–$BCEF$.	8. Why?
9. Hemisphere A–$BCEF$ = lune $\angle 180°$.	9. § 359.

The rest of the proof is left for you.

§ 362. COROLLARY. **The area of a spherical polygon is equal to the area of a lune whose angle is half the spherical excess of the polygon.**

SUGGESTION. Draw all the diagonals from one vertex.

EXERCISES

1. Find the spherical excesses of triangles whose angles are as follows:

(*a*) 80°, 95°, 103°
(*b*) 52°, 90°, 117°
(*c*) 62° 10′, 81° 40′, 96° 34′
(*d*) 56° 8′, 79° 32′, 118° 50′

2. If the spherical excess of a triangle is 60°, what is the angle of the equivalent lune? What part of the area of the sphere is the area of the triangle?

3. State the angles of lunes equivalent to triangles whose spherical excesses are as follows:

(*a*) 54°
(*b*) 115°
(*c*) 110° 30′
(*d*) $2\,a°$
(*e*) $a°$
(*f*) $a° + b°$

4. What part of the area of the sphere is the area of each of the triangles in Exercise 3?

5. State the angles of lunes equivalent to triangles whose angles are as follows:

(*a*) 75°, 95°, 120°
(*b*) 105°, 98°, 84°
(*c*) 92°, 84°, 73°
(*d*) 65°, 115°, 85°

6. What part of the area of the sphere is the area of each of the triangles in Exercise 5?

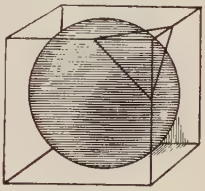
7. If the radius of the sphere is 12 inches, what is the area of each of the triangles in Exercise 5?

8. What is the area of a trirectangular triangle on a sphere whose diameter is 24 inches?

9. If the sides of a spherical triangle are 72°, 98°, and 100° on a sphere whose radius is 18 inches, what is the area of its polar triangle?

Spherical Volumes

§ 363. (1) Circumscribe a cube about a sphere whose radius is r. Then pass planes tangent to the sphere and cutting off pyramids at each vertex of the cube. The sphere is then inscribed in another polyhedron whose volume is less than the volume of the cube.



(2) Pass planes tangent to the sphere and cutting off pyramids at each vertex of this second polyhedron. Imagine this process continued indefinitely. It is evident that the volumes of these circumscribed polyhedrons become less and less as the number of faces increases.

We shall assume that the volumes of these polyhedrons approach a definite limit, and also that the sum of the areas of the faces of the polyhedrons approaches as a limit the area of the spherical surface.

§ 364. The *volume of a sphere* is the limit of the volumes of the polyhedrons described in § 363.

§ 365. Suppose that the process described in § 363 has been carried on repeatedly. Consider the pyramid formed by one of the tangent faces S_1 and the lines drawn from its vertices to the center of the sphere. The altitude of this pyramid will be equal to the radius r of the sphere. Why? Its volume then is $\frac{1}{3} S_1 r$. Similarly, we may form pyramids, using all the other tangent faces S_2, S_3, S_4, etc., as bases. Their volumes, respectively, will be $\frac{1}{3} S_2 r$, $\frac{1}{3} S_3 r$, $\frac{1}{3} S_4 r$, etc., and the sum of the volumes will be $\frac{1}{3} r(S_1+S_2+S_3+S_4+\cdots)$. This volume will approximate more and more closely the volume of the sphere as the number of faces of the polyhedron is increased. Also the sum $(S_1 + S_2 + S_3 + S_4 + \cdots)$ will approach the area of the sphere as a limit.

The discussion on the preceding page leads us to assume the following statement:

§ 366. POSTULATE. **The volume of a sphere is equal to its area multiplied by one third of its radius.**

§ 367. COROLLARY. **The volume V of a sphere whose radius is r is given by the formula**

$$V = \tfrac{4}{3}\pi r^3.$$

§ 368. COROLLARY. **The volumes of two spheres are to each other as the cubes of their radii.**

EXERCISES

1. Find the volume of a sphere whose radius is: 6 in.; 9 in.; $5\frac{1}{2}$ in.; $6\frac{7}{8}$ in.; 32.4 ft.; 4.62 ft.

2. Solve $V = \frac{4}{3}\pi r^3$ for r.

3. If the volume of a sphere is $\frac{32}{3}\pi$ cubic inches, what is its radius?

4. If the volume of a sphere is 523.3 cubic inches, what is its radius?

5. If the circumference of a great circle of a sphere is 18.9 inches, what is the volume of the sphere?

6. What must be the diameter of a hemispherical container if it is to contain 2 gallons of water?

7. Find the volume of a spherical shell whose inner and outer diameters are 10 inches and 18 inches.

8. The outer diameter of a spherical shell $\frac{3}{4}$ inch thick is 6 inches. Find its volume.

9. If the radius of a sphere is trebled, in what ratio is the volume increased?

10. What is the ratio of the radii of two spheres if the volume of one is three times the volume of the other?

11. The altitude of a right circular cylinder inscribed in a sphere of radius 8 inches is equal to its diameter. What is its lateral area? Find the ratio of its volume to the volume of the sphere.

Spherical Pyramids, Segments and Sectors

§ 369. A ***spherical pyramid*** is a portion of a sphere bounded by a spherical polygon and the planes of the great circles forming the sides of the polygon. The center of the sphere is called the ***vertex of the spherical pyramid***, and the spherical polygon is called the ***base***. *O–ABCD* in Figure 1 is a spherical pyramid with vertex *O* and the spherical polygon *ABCD* as base.

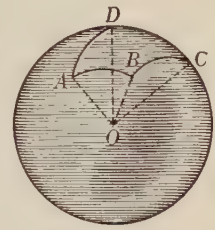


FIG. 1

§ 370. The portion of a sphere included between two parallel planes is called a ***spherical segment.*** If one of the planes is tangent to the sphere, the segment is called a ***spherical segment of one base.*** What is the distinction between a spherical segment and a spherical zone?

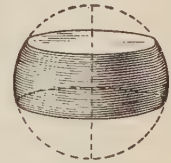


FIG. 2

§ 371. A solid generated by the revolution of a sector of a circle about any diameter of the circle is called a ***spherical sector.*** The zone generated by the arc of the sector is called the ***base of the spherical sector.*** The base of the sector may be a zone of two bases or a zone of one base, according to the position of the generating sector. A spherical sector that has for a base a zone of one base is often called a ***spherical cone.***

§ 372. POSTULATE. **The volume of a spherical sector is equal to the area of its base multiplied by one third of the radius of the sphere.** (See § 365.)

If the area of the base is S and the radius of the sphere is r, the volume V is given by the formula

$$V = \tfrac{1}{3}\, rS.$$

EXERCISES

1. Find the volume of a spherical sector the area of whose base is 30 square inches on a sphere whose radius is 5 inches.

2. Find the volume of a spherical sector whose base is a zone 3 inches high on a sphere whose radius is 10 inches.

3. A spherical sector whose base is a zone of one base is generated by a plane sector whose angle is 60° in a circle of radius 8 inches. What is the volume of the sector?

4. Explain a method of finding the volume of a segment of one base.

5. Find the volume of a segment of one base whose altitude is 2 inches on a sphere whose radius is 10 inches.

6. Make a list of all the formulas in this chapter.

7. Make a list of all the new terms in this chapter and be able to define them.

8. Make a list of the properties of the following:

(*a*) Great circles (*c*) Spherical triangles
(*b*) Small circles (*d*) Polar triangles

9. Write a formula for the area of a triangle whose spherical excess is $n°$ on a sphere whose radius is r.

10. On a sphere whose radius is 6 inches, find the areas of polygons whose angles are as follows:

(*a*) 83°, 92°, 139°, 64° (*b*) 87°, 123°, 144°, 78°

11. If a sphere is inscribed in a cylinder of revolution (that is, is tangent to its elements and bases), how does the surface of the sphere compare with the lateral surface of the cylinder?

12. How many miles is one degree measured along the 60th parallel? Assume the radius of the earth to be 4000 miles.

13. A spherical ball is placed in a conical container whose cross section is an equilateral triangle 4 inches on a side. If the ball just touches a piece of cardboard placed over the top of the container, what is the volume of the cone not occupied by the sphere?

Maintaining Skills

(*Quadratic equations*)

Any quadratic equation in one unknown may be represented by the general equation $ax^2 + bx + c = 0$, where a, b, and c are known quantities and x is unknown. If $ax^2 + bx + c$ is factorable, the solution is simple and the roots are rational. If it cannot be factored, the roots are irrational and are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 1. Solve $x^2 - 5x - 14 = 0$
Factoring, we have $(x - 7)(x + 2) = 0$
Then $x - 7 = 0$ or $x + 2 = 0$ and $x = 7$ or -2

EXAMPLE 2. Solve $3x^2 - 4x = 6$
Change first to the form $3x^2 - 4x - 6 = 0$
Then, $a = 3$, $b = -4$, $c = -6$, and

$$x = \frac{4 \pm \sqrt{16 + 72}}{6} = \frac{4 \pm \sqrt{88}}{6} = \frac{4 \pm 2\sqrt{22}}{6} = \frac{2 \pm \sqrt{22}}{3}$$

Solve the following equations:

1. $x^2 - 6x + 8 = 0$
2. $x^2 + 6x + 9 = 0$
3. $x^2 + 5x - 24 = 0$
4. $x^2 - 10x + 25 = 0$
5. $2x^2 - 5x + 6 = 0$
6. $2x^2 + x - 6 = 0$
7. $3x^2 + x - 2 = 0$
8. $6x^2 - 13x + 6 = 0$
9. $6x^2 - 13x + 5 = 0$
10. $x^2 - 3x - 7 = 0$
11. $3x^2 + 5x - 1 = 0$
12. $7x^2 - 5x - 3 = 0$
13. $5(x - 2) + x(3x - 2) = 0$
14. $(x - 2)^2 = 2(x - 4)(x + 2)$
15. $\frac{2}{x + 1} - \frac{3}{2x - 1} = 2$
16. $x^2 + 2rx + r^2 - s^2 = 0$
17. $T = \pi r(r + l)$. Solve for r.

Chapter Test

Complete the following statements:

1. AB is an arc of a great circle; PA and PB are quadrants. If arc AB is 40°, then spherical angle APB is ___?___°.

2. ABC and $A'B'C'$ are polar triangles. A is the pole of $B'C'$. If the spherical angle A is 80°, arc $B'C'$ is ___?___°.

3. The sum of the angles of a spherical triangle is greater than ___?___° and less than ___?___°.

4. If one side of a birectangular triangle is 50°, the opposite angle is ___?___°.

5. If the radius of a sphere is 5 inches, the area of the sphere is ___?___ π square inches and the volume is ___?___ π cubic inches.

6. If the altitude of a zone on a sphere whose radius is 10 inches is 2 inches, the area of the zone is ___?___ π square inches.

7. On a sphere whose area is 36 π square inches, the area of a lune whose angle is 60° is ___?___ π square inches.

8. The spherical excess of a triangle whose angles are 100°, 125°, and 140° is ___?___°.

9. The angle of a lune equivalent to a triangle whose angles are 90°, 100°, and 108° is ___?___°.

10. The spherical excess of a quadrilateral whose angles are 90°, 90°, 60°, and 150° is ___?___°.

11. The angle of a lune equivalent to the quadrilateral in the preceding exercise is ___?___°.

12. If the sides of a spherical triangle are 75°, 57°, and 112°, the smallest angle of its polar triangle is ___?___°.

13. The radius of a sphere having an area three times that of a sphere of radius a is ___?___.

14. The radius of a sphere having a volume three times that of a sphere of radius a is ___?___.

Miscellaneous Exercises

1. Prove: If from a point without a sphere a tangent and a secant are drawn, the square of the tangent is equal to the product of the whole secant and its external segment.

2. On a sphere whose diameter is 16 inches, the angles of a triangle are 75°, 63.5°, and 95.5°. What is the area of the triangle?

3. The radii of the bases of a zone are 6 inches and 10 inches. Its altitude is 5 inches. Find its area.

4. What part of the earth's surface is included between the circles of longitude 30° East and 30° West?

5. Two points A and B are 12 inches apart. Find the length of the locus of points 9 inches from A and 5 inches from B.

6. The radii of two spheres are 3 inches and 4 inches, and their centers are 5 inches apart. Find that part of the surface of each sphere which is not within the other sphere.

7. Find the length of the radius of a sphere whose volume is half that of a sphere whose radius is r.

8. What is the radius of a sphere which is equal in volume to the sum of two spheres whose radii are 2 inches and 3 inches?

9. Prove: The bisector of the vertical angle of an isosceles spherical triangle makes right angles with the base.

10. Prove: Equal chords of a small circle of a sphere are equally distant from either pole of the circle.

11. What is the ratio of the volume of a sphere to the circumscribed cube?

12. The volume of one sphere is double that of another. What is the ratio of their areas?

13. Find the area and the volume of a sphere circumscribed about a cube each of whose edges is 4 inches.

14. Prove: If any number of lines in space meet at a point, the feet of the perpendiculars drawn to these lines from any other point lie on a spherical surface.

15. When an aviator is 2 miles above the earth, how many square miles of earth may he see?

16. If the radius of a small circle of a sphere is 10 inches and its polar distance is 60°, what is the diameter of the sphere?

17. The equal sides of an isosceles spherical triangle whose base angles are each 80° are extended to form a lune. The area of the triangle is one fourth the area of the lune. Find the number of degrees in the third angle of the triangle. (Let the angle be $a°$ and solve algebraically.)

18. Prove: The three medians of a spherical triangle meet at a point.

19. Show how you would pass a plane tangent to a given sphere through a point outside the sphere.

20. Prove: If two spheres are tangent to a plane at the same point, their line of centers passes through the point of tangency.

21. Two tangent spheres whose radii are 10 inches and 4 inches rest on a horizontal table. How far apart on the table are the two points of tangency?

22. What is the altitude of a zone one of whose bases is a great circle on a sphere whose radius is 10 inches if the distance on the sphere between the two bases is 30°?

23. Three spherical balls each of whose radii is 5 inches are placed tangent to each other on a table. On top of these and tangent to the three is placed another ball of the same size. Find the distance from the table to the top of the fourth ball.

24. A cylindrical hole whose diameter is 4 inches is bored through a solid sphere whose radius is 16 inches. The axis of the cylinder coincides with a diameter of the sphere. What is the volume of the remaining solid?

25. A segment is to be cut from a sphere 10 inches in diameter so that the remaining solid will fit exactly on top of a cylindrical post 6 inches in diameter. How far from the center of the sphere should the cut be made?

26. How far from a sphere whose radius is 9 inches must a point of light be placed so that it will illuminate one third of the surface of the sphere?

27. If a cube and a sphere have equal volumes, which has the smaller area?

28. What is the volume of a spherical segment of two bases whose radii are 5 inches and 12 inches if the altitude of the segment is 7 inches?

29. A spherical ball 8 inches in diameter is dropped into an inverted tin cone. If a section of the cone through its axis is an isosceles triangle whose vertex angle is 60°, what is the circumference of the circle common to the sphere and the cone?

(Ex. 29)

30. A triangular box is to be made to contain three spherical balls, each 1 inch in diameter. Will it be more economical to make a box in which the balls may be placed one on top of the other or one in which all the balls may rest on the bottom? (Make no allowance for overlapping.)

31. The altitude of a right circular cone is 10 inches, and the radius of its base is 5 inches. It is placed on the same horizontal plane with a sphere whose radius is 5 inches. How far from the given plane must a plane be passed parallel to it to cut equal sections in the sphere and the cone?

32. The volume of a polyhedron circumscribed about a sphere whose radius is 4 inches is 500 cubic inches. What is the total area of the polyhedron?

33. The total area of a polyhedron circumscribed about a sphere whose radius is 5 inches is 560 square inches. What is the volume of the polyhedron?

34. It can be shown that the altitude of the cylinder of greatest volume which can be inscribed in a cone of revolution is one third of the altitude of the cone. If the altitude of the cone is 10 inches and the radius of the base is 7 inches, what are the dimensions of the cylinder? If the altitude of the cone is h and the radius of the base $2\ a$, what are the dimensions of the cylinder?

(Ex. 34)

35. In order to be certain that a flagpole was vertical, workmen sighted it along two plumb lines placed so that the plumb lines and the pole were not all in the same plane. Upon what geometry theorems does this method depend?

36. Find the volume of a regular octahedron whose edge is a units.

37. What is the locus of the middle points of lines joining two skew lines?

38. What is the locus of a point 10 inches from a given line l and 6 inches from a plane containing the line l?

39. What kind of solid is formed by connecting the midpoints of all the edges of a regular tetrahedron by all possible straight lines which lie in the faces of the tetrahedron?

40. Prove: The diagonals of a parallelepiped divide the solid into six pyramids which are equal in volume.

41. The total area of a regular hexagonal prism is 400 square feet. The altitude of the prism is twice the length of a side of the hexagon. Find the length of the altitude.

42. All the corners of a cube whose edge is 8 inches are cut off by planes determined by the middle points of the three edges meeting at each vertex. Describe the resulting solid, stating the shape of the faces, the number of edges, and the number of vertices. Find its total area and volume.

43. What is the volume of a sphere inscribed in a regular tetrahedron, each edge of which is 10 inches? (See Exercise 1, page 201.)

44. In order to give the most air space, a tent in the shape of a triangular prism must be pitched so that the angle at the ridge is a right angle. Find the greatest amount of air space possible in such a tent, open at both ends, made from a rectangular piece of canvas 12 feet long and 8 feet wide.

45. Prove: If the centers of the faces of a cube are joined by all possible straight lines, a regular octahedron is formed.

46. V–ABC is a regular tetrahedron. VO is perpendicular to plane ABC at O. CO meets AB at D. Prove that $\angle VDC$ is the plane angle of its dihedral angle.

47. A pyramid is formed by using four equilateral triangles. A line is drawn from the vertex to the point of intersection of the diagonals of the base. Prove that this line is perpendicular to the plane of the base.

48. Three lines not in the same plane meet at a point. Show how to draw a line which will make equal angles with these three lines.

49. Prove: If the two dihedral angles formed by one plane meeting another are bisected, the bisecting planes are perpendicular to each other.

50. Is it possible to cut the faces of a tetrahedral angle by a plane so that the intersections form a parallelogram?

51. Prove: If the projections of a line upon two intersecting planes are parallel to the intersection, the line is parallel to the intersection.

52. The ratio of the lengths of two parallel lines is $m : n$. Show that the ratio of their projections on a given plane not perpendicular to the lines is also $m : n$.

53. What is the locus of points equally distant from two skew lines? Prove that your statement is correct.

54. A sphere and a right circular cone, both on the same horizontal plane, touch each other. The radius of the sphere and the radius of the base of the cone are each 10 inches, and the altitude of the cone is 20 inches. How far apart are the point of support of the sphere and the center of the base of the cone?

SYMBOLS

$=$	is equal to, equals	$=$	equals in area or volume
$\neq$	is not equal to	$>$	is greater than
$<$	is less than	$\angle$	angle
$\triangle$	triangle	$\parallel$	is parallel to, parallel
$\perp$	is perpendicular to, perpendicular	$\odot$	circle
$\cong$	is congruent to	$\sim$	is similar to

The plural of any symbol representing a noun may be formed by adding *s*; for example, $\measuredangle$ means *angles*.

LISTS OF AXIOMS

1. Quantities equal to the same quantity are equal to each other. Quantities equal to equal quantities are equal to each other.

2. A quantity may be substituted for an equal quantity in any expression without changing the value of the expression.

3. If equal quantities are added to equal quantities, the sums are equal quantities.

4. If equal quantities are subtracted from equal quantities, the remainders are equal quantities.

5. If equal quantities are multiplied by equal quantities, the products are equal quantities. (Doubles of equals are equal.)

6. Like powers or like roots of equal quantities are equal.

7. If equal quantities are divided by equal quantities, the quotients are equal quantities. (Halves of equals are equal.)

8. If equal quantities are added to unequal quantities, the sums are unequal in the same order.

9. If equal quantities are subtracted from unequal quantities, the remainders are unequal in the same order.

10. If unequal quantities are subtracted from equal quantities, the remainders are unequal in the opposite order.

11. If unequal quantities are added to unequal quantities, the smaller to the smaller and the larger to the larger, the sums are unequal in the same order.

12. If unequal quantities are multiplied by positive equal quantities, the products are unequal in the same order. (Doubles of unequals are unequal in the same order.)

13. If unequal quantities are divided by positive equal quantities, the quotients are unequal in the same order. (Halves of unequals are unequal in the same order.)

14. The whole is greater than any of its parts.

15. The whole is equal to the sum of its parts.

16. If three quantities are so related that the first is greater than the second and the second is greater than the third, the first is greater than the third.

DEFINITIONS USED IN PLANE GEOMETRY

Acute angle. An acute angle is an angle less than a right angle.

Adjacent angles. Adjacent angles are those which have a common vertex and a common side between them.

Altitude of a triangle. An altitude of a triangle is a line from any vertex perpendicular to the opposite side, extended if necessary.

Apothem. The apothem of a regular polygon is the radius of the inscribed circle drawn to the point of tangency.

Arc. An arc is any portion of a circle. One larger than a semicircle is called a major arc; one smaller than a semicircle is called a minor arc.

Area. Area is the measure of a surface, the number of square units contained in it.

Broken line. A broken line is a succession of different straight lines.

Center of a regular polygon. The center of a regular polygon is the common center of the inscribed and circumscribed circles.

Central angle of a circle. A central angle of a circle is an angle whose vertex is at the center of the circle and whose sides are radii.

Central angle of a regular polygon. The central angle of a regular polygon is the angle between two radii drawn to the extremities of a side.

Chord. A chord is a line connecting any two points on a circle.

Circle. A circle is a closed curve all points of which are in the same plane and equally distant from a point in the plane. The point is called the **center** of the circle.

Circumference. The circumference of a circle is the length of the circle.

Circumscribed polygon. A polygon is said to be circumscribed about a circle if all of its sides are tangent to the circle. In such a case, the circle is said to be **inscribed in the polygon.**

Complementary angles. Complementary angles are two angles whose sum is equal to 90°. Each angle is the complement of the other.

Congruent figures. Congruent figures are figures which can be made to coincide. Their corresponding parts are equal.

Decagon. A decagon is a polygon of ten sides.

Diagonal. The diagonal of a polygon is a line connecting any two vertices which are not adjacent.

Diameter. A diameter of a circle is a line through the center with its extremities on the circle.

Distance from a point to a line. The distance from a point to a line is the length of the perpendicular from the point to the line.

Equiangular triangle. An equiangular triangle is one with all three angles equal.

Equilateral triangle. An equilateral triangle is one with all three sides equal.

Hexagon. A hexagon is a polygon of six sides.

Hypotenuse. The hypotenuse of a right triangle is the side opposite the right angle. The other two sides are the **arms.**

Inscribed angle. An inscribed angle is an angle whose vertex is on a circle and whose sides are chords of the circle.

Inscribed polygon. A polygon is said to be inscribed in a circle if all the vertices lie on the circle. In such a case the circle is said to be **circumscribed about the polygon.**

Isosceles trapezoid. An isosceles trapezoid is one in which the nonparallel sides are equal.

Isosceles triangle. An isosceles triangle is one with two equal sides. The two equal sides are called the arms.

Median of a trapezoid. The median of a trapezoid is a line connecting the middle points of the nonparallel sides.

Median of a triangle. A median of a triangle is a line from any vertex to the middle point of the opposite side.

Obtuse angle. An obtuse angle is an angle greater than a right angle but less than a straight angle.

Octagon. An octagon is a polygon of eight sides.

Parallel lines. Parallel lines are lines which lie in the same plane and will not meet, no matter how far extended in either direction.

Parallelogram. A parallelogram is a quadrilateral with its opposite sides parallel.

Pentagon. A pentagon is a polygon of five sides.

Perimeter. The perimeter of a polygon is the sum of the lengths of the sides.

Perpendicular lines. Perpendicular lines are lines which meet so as to form equal adjacent angles.

Polygon. A polygon is a closed broken line.

Proportion. A proportion is an equation which states that two ratios are equal.

Quadrilateral. A quadrilateral is a polygon of four sides.

Radius of a circle. A radius of a circle is a line joining the center with any point on the circle.

Radius of a regular polygon. The radius of a regular polygon is the radius of the circumscribed circle.

Ratio. A ratio is a fraction, one number divided by another.

Rectangle. A rectangle is a parallelogram with one right angle.

Regular polygon. A regular polygon is one which is both equilateral and equiangular.

Rhombus. A rhombus is a parallelogram with two adjacent sides equal.

Right angle. A right angle is one of the equal adjacent angles formed when one line is perpendicular to another.

Right triangle. A right triangle is a triangle one of whose angles is a right angle.

Secant. A secant of a circle is a line intersecting the circle in two points.

Sector. A sector of a circle is a figure formed by two radii and the intercepted arc.

Segment. A segment of a circle is a figure formed by an arc and its chord.

Semicircle. A semicircle is one half of a circle.

Similar polygons. Similar polygons are those whose corresponding angles are equal and whose corresponding sides are in proportion.

Square. A square is a rectangle with two adjacent sides equal.

Supplementary angles. Supplementary angles are two angles whose sum is equal to 180°. Each angle is the supplement of the other.

Tangent. A tangent to a circle is a line which touches the circle at only one point and does not intersect it if extended. The point where it touches the circle is called the **point of tangency** or the **point of contact.** Circles tangent to the same line at the same point are tangent to each other.

Trapezoid. A trapezoid is a quadrilateral with two and only two sides parallel. The parallel sides are called **bases of the trapezoid.** The perpendicular distance between the parallel bases is called the **altitude of the trapezoid.**

Triangle. A triangle is a polygon of three sides.

Vertical angles. Vertical angles are the opposite angles formed by two intersecting straight lines.

SYLLABUS OF PROPOSITIONS IN PLANE GEOMETRY[1]

Postulates and Preliminary Propositions

1. Any geometric figure may be moved from one place to another without changing its size or shape.

2. Between two points one and only one straight line can be drawn.

3. Two straight lines can intersect at only one point.

4. At a point in a line one and only one perpendicular can be drawn to the line (in a given plane).

5. From a point outside a line one and only one perpendicular can be drawn to the line.

6. The perpendicular is the shortest distance from a point to a line.

7. Through a point outside a line one and only one line can be drawn parallel to the line.

8. When one straight line meets another so as to form adjacent angles, the angles are supplementary.

9. If two adjacent angles form a right angle, they are complementary.

10. The exterior sides of two supplementary adjacent angles lie in the same straight line.

11. If two angles are supplementary to the same angle, or to equal angles, they are equal.

12. If two angles are complementary to the same angle, or to equal angles, they are equal.

13. When two straight lines intersect, the vertical angles are equal.

14. All right angles are equal.

15. The sum of all the successive adjacent angles formed at a point in a plane is equal to 360°.

16. The sum of all the successive adjacent angles formed at a point on a straight line, on the same side of the line, in a plane is equal to 180°.

[1] The propositions are arranged for ease of reference without regard to logical sequence.

Rectilinear Figures

Triangles

17. If two triangles have two sides and the included angle of one equal, respectively, to two sides and the included angle of the other, they are congruent.

18. If two triangles have two angles and a side of one equal, respectively, to two angles and the corresponding side of the other, they are congruent.

19. Two triangles are congruent if the three sides of one are equal, respectively, to the three sides of the other.

20. If two right triangles have the hypotenuse and an acute angle of one equal, respectively, to the hypotenuse and an acute angle of the other, they are congruent.

21. If two right triangles have the hypotenuse and another side of one equal, respectively, to the hypotenuse and a side of the other, they are congruent.

22. If two sides of a triangle are equal, the angles opposite the equal sides are equal.

23. An equilateral triangle is also equiangular.

24. If two angles of a triangle are equal, the sides opposite these angles are equal.

25. An equiangular triangle is also equilateral.

26. In a right triangle, if one angle is 30°, the hypotenuse is twice the side opposite the 30° angle.

27. In a right triangle, if the hypotenuse is twice one of the arms, the angle opposite that arm is 30°.

Parallels and Perpendiculars

28. The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

29. The line that connects the vertex of an isosceles triangle with the middle point of the base bisects the vertex angle and is perpendicular to the base.

30. Two points each equally distant from the extremities of a line determine the perpendicular bisector of the line.

31. Two lines perpendicular to a third line all in the same plane are parallel.

32. Parallel lines are at all points the same distance apart.

33. If two straight lines are cut by a transversal so that two alternate interior angles are equal, the lines are parallel.

34. When two straight lines are cut by a transversal so that two corresponding angles are equal, the lines are parallel.

35. When two straight lines are cut by a transversal so that the interior angles on the same side of the transversal are supplementary, the lines are parallel.

36. If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.

37. When two parallel lines are cut by a transversal, the alternate interior angles are equal.

38. When two parallel lines are cut by a transversal, the corresponding angles are equal.

39. When two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.

40. Two straight lines parallel to the same straight line are parallel to each other.

41. If three or more parallel lines cut off equal segments on one transversal, they cut off equal segments on every transversal.

42. The line which joins the midpoints of two sides of a triangle is parallel to the third side and equal to one half of it.

43. The median of a trapezoid is parallel to the bases and equal to one half of their sum.

44. The line which is parallel to the bases of a trapezoid and bisects one side bisects the other side also.

45. The line which is parallel to the base of a triangle and bisects one side bisects the other side also.

Parallelograms

46. A diagonal divides a parallelogram into two congruent triangles.

47. The opposite sides and the opposite angles of a parallelogram are equal.

48. The successive angles of a parallelogram are supplementary.

49. The diagonals of a parallelogram bisect each other.

50. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

51. If two sides of a quadrilateral are parallel and equal, the figure is a parallelogram.

52. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

Angles of a Polygon

53. The sum of the angles of a triangle is equal to 180°.

54. If one side of a triangle is extended, the exterior angle thus formed is equal to the sum of the two remote interior angles.

55. If two angles of one triangle are equal to two angles of another triangle, the third angles are equal.

56. Each angle of an equilateral triangle is 60°.

57. A triangle can have but one obtuse angle or one right angle.

58. Each of the acute angles of an isosceles right triangle is 45°.

59. The sum of the angles of any convex polygon of n sides is $(n - 2)\ 180°$.

60. If the angles of a polygon of n sides are equal, each angle equals $\frac{(n - 2)\ 180°}{n}$.

61. The sum of the exterior angles of any convex polygon made by extending each of the sides in succession is 360°.

Inequalities

62. The exterior angle of a triangle is greater than either remote interior angle.

63. The sum of two sides of a triangle is greater than the third side.

64. If two sides of a triangle are unequal, the angles opposite those sides are unequal in the same order.

65. If two angles of a triangle are unequal, the sides opposite those angles are unequal in the same order.

66. If in two triangles two sides of one are equal to two sides of the other and the included angle of the first is greater than the included angle of the second, the third side of the first is greater than the third side of the second.

67. If in two triangles two sides of one are equal to two sides of the other and the third side of the first is greater than the third side of the second, the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

Lines through a Point

68. The bisectors of the angles of a triangle meet at a point which is equally distant from the sides of the triangle.

69. The perpendicular bisectors of the sides of a triangle meet at a point which is equally distant from the vertices of the triangle.

70. The altitudes of a triangle meet at a point.

71. The medians of a triangle meet at a point which is two thirds of the distance from each vertex to the middle point of the opposite side.

The Circle

72. Through any three given points not lying in a straight line one circle and only one can be drawn.

73. A diameter is twice a radius.

74. All radii and all diameters of the same circle or of equal circles are respectively equal.

75. Two circles are equal if their radii or their diameters are equal.

76. A diameter bisects the circle.

77. If a line bisects a circle, it is a diameter.

78. A point is within, on, or without a circle according as its distance from the center is less than, equal to, or greater than a radius.

79. In the same circle or in equal circles equal central angles have equal arcs.

80. In the same circle or in equal circles equal arcs have equal central angles.

81. In the same circle or in equal circles equal chords have equal arcs.

82. In the same circle or in equal circles equal arcs have equal chords.

83. If a line through the center of a circle is perpendicular to a chord, it bisects the chord and its arc.

84. A line through the center of a circle that bisects a chord (not a diameter) is perpendicular to the chord.

85. The perpendicular bisector of a chord passes through the center of the circle.

86. In the same circle or in equal circles equal chords are equally distant from the center.

87. In the same circle or in equal circles chords equally distant from the center are equal.

88. In the same circle or in equal circles the larger of two central angles has the longer arc; and, conversely, the longer arc has the larger central angle.

89. In the same circle or in equal circles the longer of two minor arcs has the longer chord.

90. In the same circle or in equal circles the longer of two chords has the longer minor arc.

91. In the same circle or in equal circles unequal chords are unequally distant from the center, the longer being the nearer.

92. In the same circle or in equal circles chords unequally distant from the center are unequal, the nearer chord being the longer.

Tangents

93. A straight line perpendicular to a radius at its outer extremity is tangent to the circle.

94. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

95. A perpendicular to a tangent at the point of contact passes through the center of the circle.

96. A perpendicular to a tangent from the center of the circle passes through the point of contact.

97. The tangents to a circle from a point outside are equal.

98. If two circles are tangent to each other, their line of centers passes through the point of contact.

Measurement of Arcs and Angles

99. A central angle has the same number of degrees as its intercepted arc.

100. An inscribed angle has half as many degrees as its intercepted arc.

101. An angle inscribed in a semicircle is a right angle.

102. Inscribed angles which intercept the same arc or equal arcs are equal.

103. An angle formed by a tangent and a chord meeting it at the point of contact has half as many degrees as its intercepted arc.

104. An angle formed by two chords intersecting within a circle has half as many degrees as the sum of the two arcs intercepted by it and the vertical angle.

105. An angle formed by two secants intersecting outside a circle has half as many degrees as the difference between the two intercepted arcs.

106. An angle formed by two tangents or a secant and a tangent intersecting outside a circle has half as many degrees as the difference between the two intercepted arcs.

107. Parallel lines intercept equal arcs on a circle.

108. The opposite angles of an inscribed quadrilateral are supplementary.

LOCUS

109. The locus in a plane of a point at a given distance from a given point is a circle with the given point as center and the given distance as radius.

110. The locus in a plane of a point equally distant from two points is the perpendicular bisector of the line connecting the two points.

111. The locus in a plane of a point at a given distance from a given line is a pair of lines, one on each side of the given line, parallel to it at the given distance.

112. The locus in a plane of a point equally distant from the sides of an angle is the bisector of the angle.

113. The locus in a plane of a point equally distant from two parallel lines is a line parallel to them and midway between them.

114. The locus in a plane of the vertex of the right angle of a right triangle with a fixed hypotenuse is a circle whose diameter is the hypotenuse.

115. The locus in a plane of the vertex of a triangle with a fixed base and a vertex angle of given magnitude is a circle, passing through the extremities of the base and any one position of the vertex.

Ratio and Proportion — Similar Polygons

Ratio and Proportion

116. In any proportion the product of the means is equal to the product of the extremes.

117. The terms of a proportion are also in proportion by inversion; that is, the second is to the first as the fourth is to the third.

118. The terms of a proportion are also in proportion by alternation; that is, the first is to the third as the second is to the fourth.

119. The terms of a proportion are also in proportion by addition; that is, the first plus the second is to the first or the second as the third plus the fourth is to the third or the fourth.

120. The terms of a proportion are also in proportion by subtraction; that is, the first minus the second is to the first or second as the third minus the fourth is to the third or the fourth.

121. If three terms of one proportion are equal to the three corresponding terms of another proportion, the remaining terms are equal.

122. If the product of two numbers is equal to the product of two other numbers, either pair may be used as the means and the other as the extremes of a proportion.

123. In a series of equal ratios the sum of any number of numerators is to the sum of the corresponding denominators as any numerator is to the corresponding denominator.

124. A line parallel to one side of a triangle and intersecting the other two sides divides them proportionally.

125. Three parallel lines cut off proportional segments on any two transversals.

126. If a line is parallel to one side of a triangle, it divides the other two sides so that either is to one of its segments as the other is to the corresponding segment.

127. If a line divides two sides of a triangle proportionally, it is parallel to the third side.

128. If a line divides two sides of a triangle so that either side is to one of its segments as the other side is to the corresponding segment, it is parallel to the third side.

129. The bisector of an angle of a triangle divides the opposite side into segments which have the same ratio as the other two sides.

Similar Polygons

130. Two triangles are similar if two angles of one are equal to two angles of the other.

131. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

132. Two triangles are similar if they have an angle of one equal to an angle of the other and the sides including the angle are in proportion.

133. Two triangles are similar if their corresponding sides are in proportion.

134. Corresponding altitudes of similar triangles have the same ratio as any two corresponding sides.

135. In any right triangle the altitude upon the hypotenuse divides the triangle into two triangles each similar to the given triangle and similar to each other.

136. The perimeters of two similar polygons have the same ratio as any two corresponding sides.

Applications

137. If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.

138. If from a point outside a circle two secants are drawn, the product of one secant and its external segment is equal to the product of the other secant and its external segment.

139. If from a point outside a circle a secant and a tangent are drawn, the product of the secant and its external segment is equal to the square of the tangent.

140. In any right triangle the altitude upon the hypotenuse is the mean proportional between the segments of the hypotenuse.

141. If the altitude upon the hypotenuse of a right triangle is drawn, either arm is the mean proportional between the whole hypotenuse and the segment of the hypotenuse adjacent to the arm.

142. In any right triangle the square of the hypotenuse is equal to the sum of the squares of the arms.

Areas of Polygons

143. The area of a rectangle is equal to the product of its base and altitude.

144. The area of a parallelogram is equal to the product of its base and altitude.

145. The area of a triangle is equal to one half the product of its base and altitude.

146. The area of a trapezoid is equal to one half the product of its altitude and the sum of its bases.

147. The areas of two similar triangles are to each other as the squares of any two corresponding sides.

148. The areas of two similar polygons are to each other as the squares of any two corresponding sides.

Regular Polygons — The Measure of the Circle

Regular Polygons

149. A circle can be circumscribed about any regular polygon.

150. A circle can be inscribed in any regular polygon.

151. The central angle of a regular polygon of n sides is equal to $360° \div n$.

152. The apothem of a regular polygon is the perpendicular bisector of its side.

153. The radius of a regular polygon bisects the angle to whose vertex it is drawn.

154. If a circle is divided into three or more equal arcs, the chords of these arcs form a regular polygon of that number of sides.

155. An equilateral polygon inscribed in a circle is a regular polygon.

156. If a circle is divided into three or more equal arcs, the tangents at the points of division form a regular circumscribed polygon of that number of sides.

157. The area of a regular polygon is equal to one half the product of its apothem and its perimeter.

158. Regular polygons of the same number of sides are similar.

159. The perimeters of two regular polygons of the same number of sides are to each other as the radii or as the apothems.

160. The areas of two regular polygons of the same number of sides are to each other as the squares of the radii or as the squares of the apothems.

The Measure of the Circle

161. If the number of sides of the regular inscribed and regular circumscribed polygons of a circle is increased without limit, their perimeters both approach the circumference of the circle as a limit.

162. The circumferences of two circles are to each other as their radii.

163. The circumference of a circle is equal to the product of twice its radius and π.

164. The limit of the areas of the regular polygons inscribed in and circumscribed about a circle, as the number of sides of the polygons is increased indefinitely, is the area of the circle.

165. The limit of the apothem of a regular inscribed polygon, as the number of sides of the polygon is increased indefinitely, is the radius of the circle.

166. The area of a circle is equal to one half the product of the circumference and the radius.

167. The area of a circle is equal to π times the square of the radius.

168. The areas of two circles are to each other as the squares of their radii.

Constructions

169. Bisect a given angle.

170. At a given point on a given line construct an angle equal to a given angle.

171. At a point in a line construct a perpendicular to the line.

172. From a point outside a line construct a perpendicular to the line.

173. Through a given point outside a line draw a parallel to the line.

174. Divide a line into any number of equal parts.

175. Construct a circle which shall pass through three given points not in a straight line.

176. At a given point on a circle construct a tangent to the circle.

177. From a point outside a circle construct a tangent to the circle.

178. A triangle may be constructed if three independent parts are given. The five fundamental combinations of three independent parts are as follows:

(1) Three sides
(2) Two sides and the included angle
(3) Two angles and the included side
(4) Two angles and a side opposite one of them
(5) Two sides and an angle opposite one of them

179. Inscribe a circle in a given triangle.

180. Construct the fourth proportional to three given lines.

181. Construct the mean proportional between two given lines.

182. Transform a polygon of any number of sides into a triangle.

183. Transform a rectangle into a square.

184. Inscribe a square in a circle.

185. Inscribe a regular hexagon in a circle.

LIST OF FORMULAS IN PLANE GEOMETRY

The sum of the interior angles of any convex polygon . . . . $(n - 2)\ 180°$

One angle of any regular polygon $\dfrac{(n - 2)\ 180°}{n}$.

Area of rectangle . . . . . $A = lw$

Area of parallelogram . . . . $A = bh$

Area of triangle . . . . . $A = \dfrac{bh}{2}$

Area of trapezoid . . . . . $A = \frac{1}{2}\, h(b + b')$

Area of regular polygon . . . $A = \frac{1}{2}\, ap$

Area of circle . . . . . $A = \pi r^2$

Circumference of circle . . . . $C = 2\,\pi r$

LIST OF FORMULAS IN SOLID GEOMETRY

Lateral Area

Cone of revolution . . . . $L = \pi rs$
Cylinder of revolution . . . $L = 2\,\pi rh$
Frustum of cone of revolution $L = \frac{1}{2}\,s(c + c')$
Frustum of regular pyramid . $L = \frac{1}{2}\,s(p + p')$
Prism . . . . . $L = ep$
Regular pyramid . . . . . $L = \frac{1}{2}\,sp$

Total Area

Cone of revolution . . . . $T = \pi r(r + s)$
Cylinder of revolution . . . $T = 2\,\pi r(r + h)$
Sphere . . . . . $S = 4\,\pi r^2$
Zone . . . . . $S = 2\,\pi rh$

Volume

Circular cone . . . . . $V = \frac{1}{3}\,\pi r^2 h$
Circular cylinder . . . . . $V = Bh$
Cube . . . . . $V = e^3$
Cylinder of revolution . . . $V = \pi r^2 h$
Frustum of circular cone . . $V = \frac{1}{3}\,\pi h(r^2 + r'^2 + rr')$
Frustum of pyramid . . . . $V = \frac{1}{3}\,h(B + B' + \sqrt{BB'})$
Prism . . . . . $V = Bh$
Prismatoid . . . . . $V = \frac{1}{6}\,h(B_1 + B_2 + 4\,M)$
Pyramid . . . . . $V = \frac{1}{3}\,Bh$
Rectangular solid . . . . . $V = lwh$
Sphere . . . . . $V = \frac{4}{3}\,\pi r^3$
Spherical sector . . . . . $V = \frac{1}{3}\,rS$

TABLES

Natural Sines, Cosines, and Tangents

ANGLE	SIN	COS	TAN	ANGLE	SIN	COS	TAN
0°	.0000	1.0000	.0000	**9°**	.1564	.9877	.1584
10′	.0029	approx.	.0029	10′	.1593	.9872	.1614
20	.0058	1.	.0058	20	.1622	.9868	.1644
30	.0087		.0087	30	.1650	.9863	.1673
40	.0116	.9999	.0116	40	.1679	.9858	.1703
50	.0145	.9999	.0145	50	.1708	.9853	.1733
1	.0175	.9998	.0175	**10**	.1736	.9848	.1763
10	.0204	.9998	.0204	10	.1765	.9843	.1793
20	.0233	.9997	.0233	20	.1794	.9838	.1823
30	.0262	.9997	.0262	30	.1822	.9833	.1853
40	.0291	.9996	.0291	40	.1851	.9827	.1883
50	.0320	.9995	.0320	50	.1880	.9822	.1914
2	.0349	.9994	.0349	**11**	.1908	.9816	.1944
10	.0378	.9993	.0378	10	.1937	.9811	.1974
20	.0407	.9992	.0407	20	.1965	.9805	.2004
30	.0436	.9990	.0437	30	.1994	.9799	.2035
40	.0465	.9989	.0466	40	.2022	.9793	.2065
50	.0494	.9988	.0495	50	.2051	.9787	.2095
3	.0523	.9986	.0524	**12**	.2079	.9781	.2126
10	.0552	.9985	.0553	10	.2108	.9775	.2156
20	.0581	.9983	.0582	20	.2136	.9769	.2186
30	.0610	.9981	.0612	30	.2164	.9763	.2217
40	.0640	.9980	.0641	40	.2193	.9757	.2247
50	.0669	.9978	.0670	50	.2221	.9750	.2278
4	.0698	.9976	.0699	**13**	.2250	.9744	.2309
10	.0727	.9974	.0729	10	.2278	.9737	.2339
20	.0756	.9971	.0758	20	.2306	.9730	.2370
30	.0785	.9969	.0787	30	.2334	.9724	.2401
40	.0814	.9967	.0816	40	.2363	.9717	.2432
50	.0843	.9964	.0846	50	.2391	.9710	.2462
5	.0872	.9962	.0875	**14**	.2419	.9703	.2493
10	.0901	.9959	.0904	10	.2447	.9696	.2524
20	.0929	.9957	.0934	20	.2476	.9689	.2555
30	.0958	.9954	.0963	30	.2504	.9681	.2586
40	.0987	.9951	.0992	40	.2532	.9674	.2617
50	.1016	.9948	.1022	50	.2560	.9667	.2648
6	.1045	.9945	.1051	**15**	.2588	.9659	.2679
10	.1074	.9942	.1080	10	.2616	.9652	.2711
20	.1103	.9939	.1110	20	.2644	.9644	.2742
30	.1132	.9936	.1139	30	.2672	.9636	.2773
40	.1161	.9932	.1169	40	.2700	.9628	.2805
50	.1190	.9929	.1198	50	.2728	.9621	.2836
7	.1219	.9925	.1228	**16**	.2756	.9613	.2867
10	.1248	.9922	.1257	10	.2784	.9605	.2899
20	.1276	.9918	.1287	20	.2812	.9596	.2931
30	.1305	.9914	.1317	30	.2840	.9588	.2962
40	.1334	.9911	.1346	40	.2868	.9580	.2994
50	.1363	.9907	.1376	50	.2896	.9572	.3026
8	.1392	.9903	.1405	**17**	.2924	.9563	.3057
10	.1421	.9899	.1435	10	.2952	.9555	.3089
20	.1449	.9894	.1465	20	.2979	.9546	.3121
30	.1478	.9890	.1495	30	.3007	.9537	.3153
40	.1507	.9886	.1524	40	.3035	.9528	.3185
50	.1536	.9881	.1554	50	.3062	.9520	.3217
9	.1564	.9877	.1584	**18**	.3090	.9511	.3249

Natural Sines, Cosines, and Tangents

ANGLE	SIN	COS	TAN	ANGLE	SIN	COS	TAN
18°	.3090	.9511	.3249	**27°**	.4540	.8910	.5095
10′	.3118	.9502	.3281	10′	.4566	.8897	.5132
20	.3145	.9492	.3314	20	.4592	.8884	.5169
30	.3173	.9483	.3346	30	.4617	.8870	.5206
40	.3201	.9474	.3378	40	.4643	.8857	.5243
50	.3228	.9465	.3411	50	.4669	.8843	.5280
19	.3256	.9455	.3443	**28**	.4695	.8829	.5317
10	.3283	.9446	.3476	10	.4720	.8816	.5354
20	.3311	.9436	.3508	20	.4746	.8802	.5392
30	.3338	.9426	.3541	30	.4772	.8788	.5430
40	.3365	.9417	.3574	40	.4797	.8774	.5467
50	.3393	.9407	.3607	50	.4823	.8760	.5505
20	.3420	.9397	.3640	**29**	.4848	.8746	.5543
10	.3448	.9387	.3673	10	.4874	.8732	.5581
20	.3475	.9377	.3706	20	.4899	.8718	.5619
30	.3502	.9367	.3739	30	.4924	.8704	.5658
40	.3529	.9356	.3772	40	.4950	.8689	.5696
50	.3557	.9346	.3805	50	.4975	.8675	.5735
21	.3584	.9336	.3839	**30**	.5000	.8660	.5774
10	.3611	.9325	.3872	10	.5025	.8646	.5812
20	.3638	.9315	.3906	20	.5050	.8631	.5851
30	.3665	.9304	.3939	30	.5075	.8616	.5890
40	.3692	.9293	.3973	40	.5100	.8601	.5930
50	.3719	.9283	.4006	50	.5125	.8587	.5969
22	.3746	.9272	.4040	**31**	.5150	.8572	.6009
10	.3773	.9261	.4074	10	.5175	.8557	.6048
20	.3800	.9250	.4108	20	.5200	.8542	.6088
30	.3827	.9239	.4142	30	.5225	.8526	.6128
40	.3854	.9228	.4176	40	.5250	.8511	.6168
50	.3881	.9216	.4210	50	.5275	.8496	.6208
23	.3907	.9205	.4245	**32**	.5299	.8480	.6249
10	.3934	.9194	.4279	10	.5324	.8465	.6289
20	.3961	.9182	.4314	20	.5348	.8450	.6330
30	.3987	.9171	.4348	30	.5373	.8434	.6371
40	.4014	.9159	.4383	40	.5398	.8418	.6412
50	.4041	.9147	.4417	50	.5422	.8403	.6453
24	.4067	.9135	.4452	**33**	.5446	.8387	.6494
10	.4094	.9124	.4487	10	.5471	.8371	.6536
20	.4120	.9112	.4522	20	.5495	.8355	.6577
30	.4147	.9100	.4557	30	.5519	.8339	.6619
40	.4173	.9088	.4592	40	.5544	.8323	.6661
50	.4200	.9075	.4628	50	.5568	.8307	.6703
25	.4226	.9063	.4663	**34**	.5592	.8290	.6745
10	.4253	.9051	.4699	10	.5616	.8274	.6788
20	.4279	.9038	.4734	20	.5640	.8258	.6830
30	.4305	.9026	.4770	30	.5664	.8241	.6873
40	.4331	.9013	.4806	40	.5688	.8225	.6916
50	.4358	.9001	.4841	50	.5712	.8208	.6959
26	.4384	.8988	.4877	**35**	.5736	.8192	.7002
10	.4410	.8975	.4913	10	.5760	.8175	.7046
20	.4436	.8962	.4950	20	.5783	.8158	.7089
30	.4462	.8949	.4986	30	.5807	.8141	.7133
40	.4488	.8936	.5022	40	.5831	.8124	.7177
50	.4514	.8923	.5059	50	.5854	.8107	.7221
27	.4540	.8910	.5095	**36**	.5878	.8090	.7265

Natural Sines, Cosines, and Tangents

ANGLE	SIN	COS	TAN	ANGLE	SIN	COS	TAN
36°	.5878	.8090	.7265	**45°**	.7071	.7071	1.0000
10′	.5901	.8073	.7310	10′	.7092	.7050	1.0058
20	.5925	.8056	.7355	20	.7112	.7030	1.0117
30	.5948	.8039	.7400	30	.7133	.7009	1.0176
40	.5972	.8021	.7445	40	.7153	.6988	1.0235
50	.5995	.8004	.7490	50	.7173	.6967	1.0295
37	.6018	.7986	.7536	**46**	.7193	.6947	1.0355
10	.6041	.7969	.7581	10	.7214	.6926	1.0416
20	.6065	.7951	.7627	20	.7234	.6905	1.0477
30	.6088	.7934	.7673	30	.7254	.6884	1.0538
40	.6111	.7916	.7720	40	.7274	.6862	1.0599
50	.6134	.7898	.7766	50	.7294	.6841	1.0661
38	.6157	.7880	.7813	**47**	.7314	.6820	1.0724
10	.6180	.7862	.7860	10	.7333	.6799	1.0786
20	.6202	.7844	.7907	20	.7353	.6777	1.0850
30	.6225	.7826	.7954	30	.7373	.6756	1.0913
40	.6248	.7808	.8002	40	.7392	.6734	1.0977
50	.6271	.7790	.8050	50	.7412	.6713	1.1041
39	.6293	.7771	.8098	**48**	.7431	.6691	1.1106
10	.6316	.7753	.8146	10	.7451	.6670	1.1171
20	.6338	.7735	.8195	20	.7470	.6648	1.1237
30	.6361	.7716	.8243	30	.7490	.6626	1.1303
40	.6383	.7698	.8292	40	.7509	.6604	1.1369
50	.6406	.7679	.8342	50	.7528	.6583	1.1436
40	.6428	.7660	.8391	**49**	.7547	.6561	1.1504
10	.6450	.7642	.8441	10	.7566	.6539	1.1571
20	.6472	.7623	.8491	20	.7585	.6517	1.1640
30	.6494	.7604	.8541	30	.7604	.6494	1.1708
40	.6517	.7585	.8591	40	.7623	.6472	1.1778
50	.6539	.7566	.8642	50	.7642	.6450	1.1847
41	.6561	.7547	.8693	**50**	.7660	.6428	1.1918
10	.6583	.7528	.8744	10	.7679	.6406	1.1988
20	.6604	.7509	.8796	20	.7698	.6383	1.2059
30	.6626	.7490	.8847	30	.7716	.6361	1.2131
40	.6648	.7470	.8899	40	.7735	.6338	1.2203
50	.6670	.7451	.8952	50	.7753	.6316	1.2276
42	.6691	.7431	.9004	**51**	.7771	.6293	1.2349
10	.6713	.7412	.9057	10	.7790	.6271	1.2423
20	.6734	.7392	.9110	20	.7808	.6248	1.2947
30	.6756	.7373	.9163	30	.7826	.6225	1.2572
40	.6777	.7353	.9217	40	.7844	.6202	1.2647
50	.6799	.7333	.9271	50	.7862	.6180	1.2723
43	.6820	.7314	.9325	**52**	.7880	.6157	1.2799
10	.6841	.7294	.9380	10	.7898	.6134	1.2876
20	.6862	.7274	.9435	20	.7916	.6111	1.2954
30	.6884	.7254	.9490	30	.7934	.6088	1.3032
40	.6905	.7234	.9545	40	.7951	.6065	1.3111
50	.6926	.7214	.9601	50	.7969	.6041	1.3190
44	.6947	.7193	.9657	**53**	.7986	.6018	1.3270
10	.6967	.7173	.9713	10	.8004	.5995	1.3351
20	.6988	.7153	.9770	20	.8021	.5972	1.3432
30	.7009	.7133	.9827	30	.8039	.5948	1.3514
40	.7030	.7112	.9884	40	.8056	.5925	1.3597
50	.7050	.7092	.9942	50	.8073	.5901	1.3680
45	.7071	.7071	1.0000	**54**	.8090	.5878	1.3764

Natural Sines, Cosines, and Tangents

ANGLE	SIN	COS	TAN	ANGLE	SIN	COS	TAN
54°	.8090	.5878	1.3764	**63°**	.8910	.4540	1.9626
10′	.8107	.5854	1.3848	10′	.8923	.4514	1.9768
20	.8124	.5831	1.3934	20	.8936	.4488	1.9912
30	.8141	.5807	1.4019	30	.8949	.4462	2.0057
40	.8158	.5783	1.4106	40	.8962	.4436	2.0204
50	.8175	.5760	1.4193	50	.8975	.4410	2.0353
55	.8192	.5736	1.4281	**64**	.8988	.4384	2.0503
10	.8208	.5712	1.4370	10	.9001	.4358	2.0655
20	.8225	.5688	1.4460	20	.9013	.4331	2.0809
30	.8241	.5664	1.4550	30	.9026	.4305	2.0965
40	.8258	.5640	1.4641	40	.9038	.4279	2.1123
50	.8274	.5616	1.4733	50	.9051	.4253	2.1283
56	.8290	.5592	1.4826	**65**	.9063	.4226	2.1445
10	.8307	.5568	1.4919	10	.9075	.4200	2.1609
20	.8323	.5544	1.5013	20	.9088	.4173	2.1775
30	.8339	.5519	1.5108	30	.9100	.4147	2.1943
40	.8355	.5495	1.5204	40	.9112	.4120	2.2113
50	.8371	.5471	1.5301	50	.9124	.4094	2.2286
57	.8387	.5446	1.5399	**66**	.9135	.4067	2.2460
10	.8403	.5422	1.5497	10	.9147	.4041	2.2637
20	.8418	.5398	1.5597	20	.9159	.4014	2.2817
30	.8434	.5373	1.5697	30	.9171	.3987	2.2998
40	.8450	.5348	1.5798	40	.9182	.3961	2.3183
50	.8465	.5324	1.5900	50	.9194	.3934	2.3369
58	.8480	.5299	1.6003	**67**	.9205	.3907	2.3559
10	.8496	.5275	1.6107	10	.9216	.3881	2.3750
20	.8511	.5250	1.6212	20	.9228	.3854	2.3945
30	.8526	.5225	1.6319	30	.9239	.3827	2.4142
40	.8542	.5200	1.6426	40	.9250	.3800	2.4342
50	.8557	.5175	1.6534	50	.9261	.3773	2.4545
59	.8572	.5150	1.6643	**68**	.9272	.3746	2.4751
10	.8587	.5125	1.6753	10	.9283	.3719	2.4960
20	.8601	.5100	1.6864	20	.9293	.3692	2.5172
30	.8616	.5075	1.6977	30	.9304	.3665	2.5386
40	.8631	.5050	1.7090	40	.9315	.3638	2.5605
50	.8646	.5025	1.7205	50	.9325	.3611	2.5826
60	.8660	.5000	1.7321	**69**	.9336	.3584	2.6051
10	.8675	.4975	1.7437	10	.9346	.3557	2.6279
20	.8689	.4950	1.7556	20	.9356	.3529	2.6511
30	.8704	.4924	1.7675	30	.9367	.3502	2.6746
40	.8718	.4899	1.7796	40	.9377	.3475	2.6985
50	.8732	.4874	1.7917	50	.9387	.3448	2.7228
61	.8746	.4848	1.8040	**70**	.9397	.3420	2.7475
10	.8760	.4823	1.8165	10	.9407	.3393	2.7725
20	.8774	.4797	1.8291	20	.9417	.3365	2.7980
30	.8788	.4772	1.8418	30	.9426	.3338	2.8239
40	.8802	.4746	1.8546	40	.9436	.3311	2.8502
50	.8816	.4720	1.8676	50	.9446	.3283	2.8770
62	.8829	.4695	1.8807	**71**	.9455	.3256	2.9042
10	.8843	.4669	1.8940	10	.9465	.3228	2.9319
20	.8857	.4643	1.9074	20	.9474	.3201	2.9600
30	.8870	.4617	1.9210	30	.9483	.3173	2.9887
40	.8884	.4592	1.9347	40	.9492	.3145	3.0178
50	.8897	.4566	1.9486	50	.9502	.3118	3.0475
63	.8910	.4540	1.9626	**72**	.9511	.3090	3.0777

Natural Sines, Cosines, and Tangents

ANGLE	SIN	COS	TAN	ANGLE	SIN	COS	TAN
72°	.9511	.3090	3.0777	**81°**	.9877	.1564	6.3138
10′	.9520	.3062	3.1084	10′	.9881	.1536	6.4348
20	.9528	.3035	3.1397	20	.9886	.1507	6.5606
30	.9537	.3007	3.1716	30	.9890	.1478	6.6912
40	.9546	.2979	3.2041	40	.9894	.1449	6.8269
50	.9555	.2952	3.2371	50	.9899	.1421	6.9682
73	.9563	.2924	3.2709	**82**	.9903	.1392	7.1154
10	.9572	.2896	3.3052	10	.9907	.1363	7.2687
20	.9580	.2868	3.3402	20	.9911	.1334	7.4287
30	.9588	.2840	3.3759	30	.9914	.1305	7.5958
40	.9596	.2812	3.4124	40	.9918	.1276	7.7704
50	.9605	.2784	3.4495	50	.9922	.1248	7.9530
74	.9613	.2756	3.4874	**83**	.9925	.1219	8.1443
10	.9621	.2728	3.5261	10	.9929	.1190	8.3450
20	.9628	.2700	3.5656	20	.9932	.1161	8.5555
30	.9636	.2672	3.6059	30	.9936	.1132	8.7769
40	.9644	.2644	3.6470	40	.9939	.1103	9.0098
50	.9652	.2616	3.6891	50	.9942	.1074	9.2553
75	.9659	.2588	3.7321	**84**	.9945	.1045	9.5144
10	.9667	.2560	3.7760	10	.9948	.1016	9.7882
20	.9674	.2532	3.8208	20	.9951	.0987	10.078
30	.9681	.2504	3.8667	30	.9954	.0958	10.385
40	.9689	.2476	3.9136	40	.9957	.0929	10.712
50	.9696	.2447	3.9617	50	.9959	.0901	11.059
76	.9703	.2419	4.0108	**85**	.9962	.0872	11.430
10	.9710	.2391	4.0611	10	.9964	.0843	11.826
20	.9717	.2363	4.1126	20	.9967	.0814	12.251
30	.9724	.2334	4.1653	30	.9969	.0785	12.706
40	.9730	.2306	4.2193	40	.9971	.0756	13.197
50	.9737	.2278	4.2747	50	.9974	.0727	13.727
77	.9744	.2250	4.3315	**86**	.9976	.0698	14.301
10	.9750	.2221	4.3897	10	.9978	.0669	14.924
20	.9757	.2193	4.4494	20	.9980	.0640	15.605
30	.9763	.2164	4.5107	30	.9981	.0610	16.350
40	.9769	.2136	4.5736	40	.9983	.0581	17.169
50	.9775	.2108	4.6382	50	.9985	.0552	18.075
78	.9781	.2079	4.7046	**87**	.9986	.0523	19.081
10	.9787	.2051	4.7729	10	.9988	.0494	20.206
20	.9793	.2022	4.8430	20	.9989	.0465	21.470
30	.9799	.1994	4.9152	30	.9990	.0436	22.904
40	.9805	.1965	4.9894	40	.9992	.0407	24.542
50	.9811	.1937	5.0658	50	.9993	.0378	26.432
79	.9816	.1908	5.1446	**88**	.9994	.0349	28.636
10	.9822	.1880	5.2257	10	.9995	.0320	31.242
20	.9827	.1851	5.3093	20	.9996	.0291	34.368
30	.9833	.1822	5.3955	30	.9997	.0262	38.188
40	.9838	.1794	5.4845	40	.9997	.0233	42.964
50	.9843	.1765	5.5764	50	.9998	.0204	49.104
80	.9848	.1736	5.6713	**89**	.9998	.0175	57.290
10	.9853	.1708	5.7694	10	.9999	.0145	68.750
20	.9858	.1679	5.8708	20	.9999	.0116	85.940
30	.9863	.1650	5.9758	30	approx. 1.	.0087	114.59
40	.9868	.1622	6.0844	40		.0058	171.89
50	.9872	.1593	6.1970	50		.0029	343.77
81	.9877	.1564	6.3138	**90**	1.0000	.0000	Infinite

Common Logarithms of Numbers[1]

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

[1] This table gives the mantissas of numbers with the decimal point omitted in each case. Characteristics are determined by inspection from the numbers.

Common Logarithms of Numbers

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Logarithms of Trigonometric Functions[1]

ANGLE	LOG SIN	LOG COS	LOG TAN	LOG COT	
1°	8.2419	9.9999	8.2419	1.7581	89°
2°	8.5428	9.9997	8.5431	1.4569	88°
3°	8.7188	9.9994	8.7194	1.2806	87°
4°	8.8436	9.9989	8.8446	1.1554	86°
5°	8.9403	9.9983	8.9420	1.0580	85°
6°	9.0192	9.9976	9.0216	0.9784	84°
7°	9.0859	9.9968	9.0891	0.9109	83°
8°	9.1436	9.9958	9.1478	0.8522	82°
9°	9.1943	9.9946	9.1997	0.8003	81°
10°	9.2397	9.9934	9.2463	0.7537	80°
11°	9.2806	9.9919	9.2887	0.7113	79°
12°	9.3179	9.9904	9.3275	0.6725	78°
13°	9.3521	9.9887	9.3634	0.6366	77°
14°	9.3837	9.9869	9.3968	0.6032	76°
15°	9.4130	9.9849	9.4281	0.5719	75°
16°	9.4403	9.9828	9.4575	0.5425	74°
17°	9.4659	9.9806	9.4853	0.5147	73°
18°	9.4900	9.9782	9.5118	0.4882	72°
19°	9.5126	9.9757	9.5370	0.4630	71°
20°	9.5341	9.9730	9.5611	0.4389	70°
21°	9.5543	9.9702	9.5842	0.4158	69°
22°	9.5736	9.9672	9.6064	0.3936	68°
23°	9.5919	9.9640	9.6279	0.3721	67°
24°	9.6093	9.9607	9.6486	0.3514	66°
25°	9.6259	9.9573	9.6687	0.3313	65°
26°	9.6418	9.9537	9.6882	0.3118	64°
27°	9.6570	9.9499	9.7072	0.2928	63°
28°	9.6716	9.9459	9.7257	0.2743	62°
29°	9.6856	9.9418	9.7438	0.2562	61°
30°	9.6990	9.9375	9.7614	0.2386	60°
31°	9.7118	9.9331	9.7788	0.2212	59°
32°	9.7242	9.9284	9.7958	0.2042	58°
33°	9.7361	9.9236	9.8125	0.1875	57°
34°	9.7476	9.9186	9.8290	0.1710	56°
35°	9.7586	9.9134	9.8452	0.1548	55°
36°	9.7692	9.9080	9.8613	0.1387	54°
37°	9.7795	9.9023	9.8771	0.1229	53°
38°	9.7893	9.8965	9.8928	0.1072	52°
39°	9.7989	9.8905	9.9084	0.0916	51°
40°	9.8081	9.8843	9.9238	0.0762	50°
41°	9.8169	9.8778	9.9392	0.0608	49°
42°	9.8255	9.8711	9.9544	0.0456	48°
43°	9.8338	9.8641	9.9697	0.0303	47°
44°	9.8418	9.8569	9.9848	0.0152	46°
45°	9.8495	9.8495	0.0000	0.0000	45°
	LOG COS	LOG SIN	LOG COT	LOG TAN	ANGLE

[1] For angles less than 45°, the logarithm of a function may be found by reading the number of degrees in the column at the left and the name of the function at the top of the column. For angles greater than 45°, read the number of degrees in the column at the *right* side of the page and read the name of the function at the *bottom* of the column.

In this table, 7 to the left of the decimal point means that the characteristic is 7–**10**; 8 means 8–**10**, etc., for any figure greater than 6.

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